Singular minimizers in the calculus of variations and nonlinear elasticity

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Award date:
2007

Awarding institution:
University of Bath

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for the degree of Doctor of Philosophy

of the

University of Bath

Department of Mathematical Sciences

April 2007

Summary

This thesis studies the $W^{1,p}$-quasiconvexity of some typical stored energy functions from nonlinear elasticity. For a homogeneous hyperelastic body which is subjected to affine boundary displacements we investigate whether there exist any (singular) deformations with less energy than that of the corresponding homogeneous deformation.

We give conditions on the set of matrices representing the homogeneous boundary displacements which are either necessary or sufficient for cavitation to be energetically favourable.

Also investigated are the problems of the optimal location of a solitary hole in an elastic body and the uniqueness of weak solutions of the equilibrium equations of nonlinear elasticity.

Acknowledgements

Firstly, I would like to thank my supervisor Dr Jey Sivaloganathan for his help and advice over the years and for giving me some interesting problems to work on.

I would also like to thank my examiners Professor Charles A. Stuart and Professor Geoffrey Burton for their thorough reading of my thesis, their appreciation of my work and their encouragements.

Many thanks also to the staff and postgraduate students in the Department of Mathematical Sciences for providing a friendly and supportive environment. Special thanks go to my nice officemates: Damien, Doku, James, Jason and Matt.

This work was supported by a Department of Mathematical Sciences Studentship and a University of Bath ORS award.

Finally, my warmest thanks go to Eugen, for his great love and support over the years.

Contents

Chapter 1

Introduction

1.1 Variational problems in Nonlinear Elasticity

A central problem in nonlinear three-dimensional elasticity is to find the equilibrium states of an elastic body which in its reference configuration occupies a bounded open connected subset Ω of $\mathbb{R}^3$ (with a Lipschitz continuous boundary) which deforms when subjected to boundary displacements or loads. A deformation of the body corresponds to a mapping $\mathbf{u} : \Omega \to \mathbb{R}^3$ that lies in the Sobolev space $W^{1,1}(\Omega; \mathbb{R}^3)$. Deformations are required to satisfy the local invertibility condition

$$\det \nabla \mathbf{u}(\mathbf{x}) > 0 \quad \text{a.e. } \mathbf{x} \in \Omega, \tag{1.1}$$

where $\nabla \mathbf{u}$ denotes the matrix of weak derivatives of $\mathbf{u}$.

When the material is **homogeneous** and **hyperelastic**, the total elastic energy stored in a body that undergoes such a deformation is given by

$$E(\mathbf{u}) = \int_\Omega W(\nabla \mathbf{u}(\mathbf{x}))\, d\mathbf{x} \tag{1.2}$$

where $W : M_+^{3\times 3} \to [0, \infty)$ is the **stored energy function** of the material, and $M_+^{3\times 3}$ denotes the set of real 3×3 matrices with positive determinant. The assumption that $W \geq 0$ is made for convenience (it is natural to assume that W is bounded below, and adding a constant does not change the problem).

It is customary in elasticity to assume that W is **frame-indifferent**, which means that the energy of a deformation is invariant under changes in observer, and is expressed mathematically by the condition

$$W(\mathbf{QF}) = W(\mathbf{F}) \quad \text{for all } \mathbf{F} \in M_+^{3\times3},\ \mathbf{Q} \in SO(3). \tag{1.3}$$

It is also assumed that W is **isotropic**, which means that the material has no preferred direction as regards its mechanical response, and is expressed by the condition

$$W(\mathbf{FQ}) = W(\mathbf{F}) \quad \text{for all } \mathbf{F} \in M_+^{3\times3},\ \mathbf{Q} \in SO(3). \tag{1.4}$$

In the above, $SO(3)$ denotes the special orthogonal group on $\mathbb{R}^3$.

Another key requirement in elasticity is that the stored energy function W, assumed to be smooth on $M_+^{3\times3}$, satisfies

$$W(\mathbf{F}) \to \infty \quad \text{as} \quad \det \mathbf{F} \to 0, \tag{1.5}$$

which mathematically reflects the idea that large energies must accompany severe compressions. Consistent with the requirement (1.1), we extend W to the set $M^{3\times3}$ of all 3×3 matrices, by setting

$$W(\mathbf{F}) = +\infty \quad \text{for all } \mathbf{F} \text{ with } \det \mathbf{F} \leq 0. \tag{1.6}$$

In this way, W is a continuous function from $M^{3\times3}$ to $[0, \infty]$.

In homogeneous hyperelasticity, the **Piola-Kirchhoff stress tensor** T_R : $M_+^{3\times3} \to M^{3\times3}$ is given by

$$T_R(\mathbf{F}) = \frac{\partial W}{\partial \mathbf{F}}(\mathbf{F}) := \left(\frac{\partial W}{\partial F_j^i}(\mathbf{F})\right) \quad \text{for all } \mathbf{F} \in M_+^{3\times3}. \tag{1.7}$$

The **Cauchy stress tensor** $T(\mathbf{F})$ is related to $T_R(\mathbf{F})$ through the formula

$$T(\mathbf{F}) = (\det \mathbf{F})^{-1} T_R(\mathbf{F})\mathbf{F}^T. \tag{1.8}$$

The tensors T_R and T measure the force on the body per unit area in the undeformed and deformed configurations respectively.

For a homogeneous hyperelastic body with stored energy function W the

equilibrium equations under zero body force are given by

$$\frac{\partial}{\partial x_\alpha}\left[\frac{\partial W}{\partial F^i_\alpha}(\nabla \mathbf{u})\right] = 0 \quad \text{for } i = 1, 2, 3, \tag{1.9}$$

where we use the convention of summation over repeated indices. These are the **Euler-Lagrange equations** for the functional E given by (1.2).

Let $\partial\Omega_1 \subset \partial\Omega$ be a portion of the boundary with $\mathcal{H}^2(\partial\Omega_1) > 0$, where $\mathcal{H}^2$ denotes two-dimensional Hausdorff measure (i.e. surface area). We consider the boundary condition

$$\mathbf{u}|_{\partial\Omega_1} = \mathbf{f}, \tag{1.10}$$

for a given mapping $\mathbf{f} : \partial\Omega_1 \to \mathbb{R}^3$, and we impose no boundary conditions on the remaining part of the boundary $\partial\Omega \setminus \partial\Omega_1$.

In the variational approach, equilibrium solutions to this mixed displacement/zero traction problem are found by minimising E over a class of admissible deformations contained in $W^{1,1}(\Omega; \mathbb{R}^3)$ and satisfying (1.1) and (1.10). Such minimisers would formally satisfy (1.9), and the corresponding applied traction would vanish on $\partial\Omega \setminus \partial\Omega_1$.

The nonlinear elasticity problem can be regarded as a particular instance of a problem in the calculus of variations, where one seeks minimisers of an energy functional

$$E(\mathbf{u}) = \int_\Omega g(\nabla \mathbf{u}(\mathbf{x}))\, d\mathbf{x}, \tag{1.11}$$

among functions $\mathbf{u} : \Omega \subset \mathbb{R}^m \to \mathbb{R}^n$ satisfying a boundary condition such as (1.10), where $g : M^{n\times m} \to [0, \infty]$ is a continuous function, and $m, n \geq 1$. A successful approach for solving this problem is provided by the **direct method of the calculus of variations**, see for example Dacorogna [16]. The essence of this method is the following. Take a minimising sequence $\{\mathbf{u}_j\}_{j\geq 1}$ of E, that is a sequence such that

$$E(\mathbf{u}_j) \to \alpha = \inf E \quad \text{as } j \to \infty. \tag{1.12}$$

Assuming (passing to a subsequence if necessary) that $\mathbf{u}_j$ converges in a certain topology to a limit $\mathbf{u}$, and that the functional E is (sequentially) lower semicontinuous with respect to that topology, i.e. $E(\mathbf{u}) \leq \liminf_{j\to\infty} E(\mathbf{u}_j)$ whenever

$\mathbf{u}_j \to \mathbf{u}$, then the existence of minimizers of E is guaranteed, since

$$\alpha \leq E(\mathbf{u}) \leq \liminf_{j\to\infty} E(\mathbf{u}_j) = \alpha. \tag{1.13}$$

The function spaces which are usually used are the Sobolev spaces $W^{1,p}(\Omega;\mathbb{R}^n)$, $1 \leq p \leq \infty$, endowed with their weak topology (weak* if $p = \infty$). If a minimizing sequence can be shown to be bounded in a space $W^{1,p}$, $1 < p \leq \infty$, then the existence of a weakly convergent subsequence (weakly* convergent if $p = \infty$) is guaranteed by the Banach-Alaoglu Theorem. Therefore it remains as a question of main interest whether the functional is sequentially weakly lower semicontinuous (swlsc) on $W^{1,p}(\Omega;\mathbb{R}^n)$, $1 \leq p < \infty$, or sequentially weak* lower semicontinuous (sw*lsc) on $W^{1,\infty}(\Omega;\mathbb{R}^n)$.

In his fundamental paper [31], Morrey introduced the notion of quasiconvexity and he showed that, for continuous, finite-valued integrands, the quasiconvexity of g is equivalent to the sw*lsc of E over $W^{1,\infty}(\Omega;\mathbb{R}^n)$. A function g is said to be **quasiconvex at $\mathbf{F} \in M^{n\times m}$** if and only if, for all non-empty open, bounded subsets $D \subset \mathbb{R}^m$,

$$\int_D g(\mathbf{F} + \nabla\varphi(\mathbf{x}))\, d\mathbf{x} \geq \int_D g(\mathbf{F})\, d\mathbf{x} \quad \text{for all } \varphi \in W_0^{1,\infty}(D;\mathbb{R}^n). \tag{1.14}$$

The function g is said to be **quasiconvex** if it is quasiconvex at every matrix $\mathbf{F} \in M^{n\times m}$. Morrey's results have been refined by a number of authors. In particular, Acerbi and Fusco [2] showed that, if g is continuous and satisfies

$$0 \leq g(\mathbf{F}) \leq K(|\mathbf{F}|^p + 1) \quad \text{for all } \mathbf{F} \in M^{n\times m}, \tag{1.15}$$

then E is swlsc on $W^{1,p}(D;\mathbb{R}^n)$ if and only if g is quasiconvex. Note, however, that all these results require the integrand g to be finite-valued, and therefore do not apply to the nonlinear elasticity setting, where (1.6) holds.

A refinement of the quasiconvexity condition was introduced and studied by Ball and Murat [11]. The new condition, called **$W^{1,p}$-quasiconvexity**, for $1 \leq p \leq \infty$, generalizes in a natural way the quasiconvexity condition by allowing the competing functions in (1.14) to belong to the Sobolev space $W_0^{1,p}(D;\mathbb{R}^n)$, rather than to the smaller space $W_0^{1,\infty}(D;\mathbb{R}^n)$. A function g is said to be **$W^{1,p}$-quasiconvex at $\mathbf{F} \in M^{n\times m}$** if and only if, for all non-empty open, bounded subsets $D \subset \mathbb{R}^m$,

$$\int_D g(\mathbf{F} + \nabla\varphi(\mathbf{x}))\, d\mathbf{x} \geq \int_D g(\mathbf{F})\, d\mathbf{x} \quad \text{for all } \varphi \in W_0^{1,p}(D;\mathbb{R}^n), \tag{1.16}$$

and is said to be **$W^{1,p}$-quasiconvex** if it is $W^{1,p}$-quasiconvex at every matrix $\mathbf{F} \in M^{n\times m}$. Ball and Murat [11] showed that the $W^{1,p}$-quasiconvexity of g is a necessary condition for E to be swlsc on $W^{1,p}(D;\mathbb{R}^n)$. They also conjectured that when $g : M^{n\times n} \to [0,\infty]$ is continuous, then $W^{1,p}$-quasiconvexity of g, or some slight variant of it, is sufficient for E to be swlsc on $W^{1,p}(D;\mathbb{R}^n)$. However, they were not able to prove this, and the question continues to remain open. These notions of quasiconvexity are of great importance in the calculus of variations.

Significant progress on the existence theory for elasticity problems was made by Ball [4]. He showed that, among deformations in $W^{1,1}(\Omega;\mathbb{R}^3)$ satisfying (1.1) and (1.10), the energy (1.2) does indeed attain its infimum, provided that the stored energy W, in addition to being sufficiently coercive, is also polyconvex. A stored energy function W is said to be **polyconvex** if there exists a function f, convex on the set $M^{3\times3} \times M^{3\times3} \times (0,\infty)$, such that

$$W(\mathbf{F}) = f(\mathbf{F},\ \operatorname{adj}\mathbf{F}, \det\mathbf{F}) \quad \text{for all } \mathbf{F} \in M_+^{3\times3}.$$

However, the coerciveness conditions required for Ball's results and many of their refinements are so strong that they are incompatible with the phenomenon of cavitation, which means the formation of new holes in the material, and which has been observed in experiments on elastomers, see Gent and Lindley [21].

The foundations for the mathematical study of cavitation were laid in a pioneering paper of Ball [8], in the radial setting. When Ω is the unit ball in $\mathbb{R}^3$, Ball carried out the minimization of the total energy E among the restricted class of radially symmetric deformations in $W^{1,1}(\Omega;\mathbb{R}^3)$ satisfying (1.1) of the form

$$\mathbf{u}(\mathbf{x}) = r(|\mathbf{x}|)\frac{\mathbf{x}}{|\mathbf{x}|} \quad \text{for some } r : [0,1] \to [0,\infty]. \tag{1.17}$$

In particular, he considered radial deformations satisfying displacement boundary conditions $\mathbf{u}(\mathbf{x}) = \lambda\mathbf{x}$ on the boundary of Ω and showed the existence of global minimizers for a large class of materials. Moreover, he showed that, for λ sufficiently large, the global minimizer is not the expected homogeneous deformation

$\mathbf{u}(\mathbf{x}) = \lambda\mathbf{x}$, but a deformation satisfying $r(0) > 0$ which creates a spherical hole at the origin.

A typical class of stored energy functions for which the above results on cavitation apply is given by

$$W_h(\mathbf{F}) = |\mathbf{F}|^p + h(\det \mathbf{F}) \quad \text{for all } \mathbf{F} \in M_+^{3\times 3}, \tag{1.18}$$

where $1 \leq p < 3$ and $h : (0, \infty) \to [0, \infty)$ is a C^1 function required to satisfy

$$h \text{ convex, } h(s) \to \infty \text{ as } s \to 0^+, \ \frac{h(s)}{s} \to \infty \text{ as } s \to \infty. \tag{1.19}$$

(Note that, to allow for discontinuous deformations it is necessary to work in Sobolev spaces $W^{1,p}(\Omega; \mathbb{R}^3)$ with $1 \leq p < 3$. Indeed, for $p > 3$ every mapping in $W^{1,p}(\Omega; \mathbb{R}^3)$ is continuous by the Sobolev Imbedding Theorem, while any mapping in $W^{1,3}(\Omega; \mathbb{R}^3)$ satisfying an additional invertibility condition is also continuous, see Proposition 1.34.)

It is a problem of great interest to give an existence theory in classes of non-radial deformations allowing for cavitation. The main difficulty is that the elastic energy functional is not swlsc if cavitation is energetically favorable. Indeed, the sequential weakly lower semicontinuity of the functional on $W^{1,p}(\Omega; \mathbb{R}^3)$ would imply that W is $W^{1,p}$-quasiconvex, which is not the case when (radial) cavitation occurs. More precisely, as pointed out in [11], if there is a $\lambda > 0$ and a map $\mathbf{u} \in W^{1,p}(\Omega; \mathbb{R}^3)$ with $\mathbf{u}(\mathbf{x}) = \lambda\mathbf{x}$ on $\partial\Omega$, where Ω is the unit ball in $\mathbb{R}^3$, such that

$$\int_\Omega W(\nabla \mathbf{u}(\mathbf{x}))\, d\mathbf{x} < \int_\Omega W(\lambda \mathbf{I})\, d\mathbf{x}, \tag{1.20}$$

then by rescaling $\mathbf{u}$ and covering Ω with small balls one can easily construct a sequence $\{\mathbf{u}_j\}_{j\geq 1}$ such that $\mathbf{u}_j \rightharpoonup \lambda\mathbf{id}$ weakly in $W^{1,p}(\Omega; \mathbb{R}^3)$, where $\mathbf{id}(\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x} \in \Omega$, and

$$\lim_{j\to\infty} \int_\Omega W(\nabla \mathbf{u}_j(\mathbf{x}))\, d\mathbf{x} = \int_\Omega W(\nabla \mathbf{u}(\mathbf{x}))\, d\mathbf{x} < \int_\Omega W(\lambda \mathbf{I})\, d\mathbf{x}. \tag{1.21}$$

The first existence theory in function spaces allowing for cavitation was given by Müller and Spector [34]. Their approach was, however, somewhat non-standard, since they did not minimize the elastic energy of the body given by (1.2) but rather a total energy functional consisting of the sum of the elastic energy and another term accounting for surface energy. A significant contribution of their work is the introduction and detailed study of a class of admissible deformations which is well suited for modelling the phenomenon of cavitation, and which has been used in further works. This class is contained in $W^{1,p}(\Omega;\mathbb{R}^3)$ for $p \in (2,3)$, so that for these mappings, which are potentially discontinuous, their restrictions to (almost every) two-dimensional surfaces are continuous, and the topological degree theory is an important tool that can be utilised in their setting. For any $\mathbf{A} \in M^{3\times 3}_+$ and $p \in (2,3)$, let

$$\begin{aligned}\mathcal{A}_{\mathbf{A},p} := \{\mathbf{u} \in W^{1,p}(\Omega;\mathbb{R}^3)\ : \mathbf{u} - \mathbf{A}\mathbf{x} \in W^{1,p}_0(\Omega;\mathbb{R}^3),\ \det \nabla\mathbf{u} > 0 \text{ a.e.},\\ \mathbf{u}^e \text{ satisfies (INV)}\}.\end{aligned} \tag{1.22}$$

Here, for every deformation $\mathbf{u}$ satisfying $\mathbf{u}(\mathbf{x}) = \mathbf{A}\mathbf{x}$ on $\partial\Omega$, one denotes by $\mathbf{u}^e$ its homogeneous extension to all of $\mathbb{R}^3$ as the linear deformation $\mathbf{A}\mathbf{x}$. Roughly speaking, the (INV) condition is the requirement that holes produced within one part of the body are not filled by material from other parts (see Definition 1.31 for a precise meaning). One merit of the (INV) condition is that it implies the invertibility almost everywhere of mappings $\mathbf{u}$ with the additional property that $\det \nabla\mathbf{u} \neq 0$ a.e. (see Proposition 1.33). Global invertibility (almost everywhere) is an important requirement for admissible deformations, since matter cannot interpenetrate itself.

An important role in the investigation in [34] is played by the distributional Jacobian. For mappings $\mathbf{u}$ in $\mathcal{A}_{\mathbf{A},p}$, this is originally defined as a distribution

$$(\mathrm{Det}\nabla\mathbf{u}^e)(\phi) := -\frac{1}{3}\int_{\mathbb{R}^3} \nabla\phi \cdot (\mathrm{adj}\nabla\mathbf{u}^e)\mathbf{u}^e\,d\mathbf{x}, \qquad \phi \in C^\infty_0(\mathbb{R}^3), \tag{1.23}$$

but it is shown there that this distribution is actually generated by a nonnegative Radon measure, of the form

$$\mathrm{Det}\,\nabla\mathbf{u}^e = (\det\nabla\mathbf{u}^e)\mathcal{L}^3 + m_{\mathbf{u}}, \tag{1.24}$$

where $m_{\mathbf{u}}$ is nonnegative and singular with respect to Lebesgue measure $\mathcal{L}^3$. Of particular interest are mappings $\mathbf{u}$ for which $m_{\mathbf{u}}$ is a finite linear combination of

Dirac measures, i.e.

$$\text{Det } \nabla \mathbf{u}^e = (\det \nabla \mathbf{u}^e)\mathcal{L}^3 + \sum_{i=1}^{N} \alpha_i \delta_{\mathbf{a}_i}, \;\; \mathbf{a}_i \in \Omega, \; \alpha_i \geq 0. \tag{1.25}$$

For example, if $\mathbf{u}$ is a radial map in $W^{1,p}(\Omega;\mathbb{R}^3)$ with $\det \nabla \mathbf{u} > 0$ almost everywhere, then $\mathbf{u}$ satisfies (INV) and the singular measure $m_{\mathbf{u}}$ in (1.24) satisfies

$$m_{\mathbf{u}} = \frac{4\pi}{3} r^3(0)\delta_{\mathbf{0}},$$

where $\delta_{\mathbf{0}}$ is the Dirac measure supported at $\mathbf{0}$. Consistent with the situation in the radial case, there is a sense (see [34]) in which deformations satisfying (1.25) can be interpreted as producing new holes (not necessarily spherical) of volume α_i at the points $\mathbf{a}_i$.

A new model for cavitation was proposed by Sivaloganathan and Spector [44], in which new holes in the material could occur only at a, possibly large, number of infinitesimal flaws. This was modelled mathematically by using admissible deformations whose possible point discontinuities are constrained to be at the specified flaw points. Namely, given a finite set of points $\mathbf{a}_i \in \Omega$, $i = 1, ..., N$, they considered the class

$$\begin{aligned} \mathcal{A}_{\mathbf{A},p}(\mathbf{a}_1, ..., \mathbf{a}_N) = \Big\{ \mathbf{u} \in \mathcal{A}_{\mathbf{A},p} : \text{ Det } \nabla \mathbf{u}^e = (\det \nabla \mathbf{u}^e)\mathcal{L}^3 + \sum_{i=1}^{N} \alpha_i \delta_{\mathbf{a}_i}, \\ \alpha_i \geq 0 \text{ for all } i = 1, ..., N \Big\} \end{aligned} \tag{1.26}$$

and, for a class of polyconvex stored energy functions W which includes those of the form (1.18), with $2 < p < 3$, they showed the existence of minimisers in $\mathcal{A}_{\mathbf{A},p}(\mathbf{a}_1, ..., \mathbf{a}_N)$, using the direct methods of the calculus of variations. They also showed in [45] that any minimiser given by this result must produce a discontinuity if the boundary displacement is sufficiently large. More precisely, they proved that, if $\mathbf{A} = t\mathbf{B}$, where $\mathbf{B} \in M_+^{3\times 3}$ is fixed and $t > 0$, then for sufficiently large t any minimiser of E on $\mathcal{A}_{\mathbf{A},p}(\mathbf{a}_1, ..., \mathbf{a}_N)$ must satisfy $\alpha_i > 0$ for some i.

For any $\mathbf{A} \in M_+^{3\times 3}$, let us denote by $\mathbf{u}_{\mathbf{A}}^{\text{hom}}$ the homogeneous deformation $\mathbf{u}_{\mathbf{A}}^{\text{hom}}(\mathbf{x}) \equiv \mathbf{A}\mathbf{x}$ in Ω. The $W^{1,p}$-quasiconvexity condition (1.16) at $\mathbf{A}$ for a stored energy function W can be reformulated as

$$E(\mathbf{u}_{\mathbf{A}}^{\mathrm{hom}}) \leq E(\mathbf{u}) \quad \text{for all } \mathbf{u} \in W^{1,p}(\Omega;\mathbb{R}^3) \text{ with } \mathbf{u}(\mathbf{x}) = \mathbf{A}\mathbf{x} \text{ on } \partial\Omega. \tag{1.27}$$

It is apparent from the previous discussion that, for any stored energy function for which cavitation is favourable, this condition fails at some matrices $\mathbf{A} \in M_+^{3\times 3}$.

The main question that we address in this thesis is, given a domain Ω and a stored energy function W, to determine the set of matrices $\mathbf{A}$ for which W is $W^{1,p}$-quasiconvex at $\mathbf{A}$, or at least to determine conditions which are either necessary or sufficient for W to be $W^{1,p}$-quasiconvex at a matrix $\mathbf{A}$. Since a simple scaling argument shows that the $W^{1,p}$-quasiconvexity of W does not depend on the domain Ω, there is no loss of generality in assuming when convenient that $\Omega := B(\mathbf{0}, 1)$, the unit ball in $\mathbb{R}^3$.

In this thesis we pursue this problem by a purely analytical approach. The results could also be of practical interest, since they may model fracture in a nonlinear elastic material.

Since, given affine boundary displacement conditions, there is more than one possible class of admissible deformations in which to seek minimisers of the energy, it is natural to consider refinements of the $W^{1,p}$-quasiconvexity condition (1.27), such as $W^{1,p}$-quasiconvexity over $\mathcal{A}_{\mathbf{A},p}$, over $\mathcal{A}_{\mathbf{A},p}(\mathbf{a}_1, ..., \mathbf{a}_N)$ or, when $\mathbf{A} = \lambda\mathbf{I}$, over the class over radial deformations. Given a stored energy function W, a matrix $\mathbf{A} \in M_+^{n\times n}$ and a class of mappings $\mathcal{A}$ contained in $W^{1,p}(\Omega;\mathbb{R}^n)$, we say that W is **$W^{1,p}$-quasiconvex at $\mathbf{A}$ over $\mathcal{A}$** if

$$E(\mathbf{u}_{\mathbf{A}}^{\mathrm{hom}}) \leq E(\mathbf{u}) \quad \text{for all } \mathbf{u} \in \mathcal{A} \text{ with } \mathbf{u}(\mathbf{x}) = \mathbf{A}\mathbf{x} \text{ on } \partial\Omega,$$

where, for all $\mathbf{u} \in \mathcal{A}$, $E(\mathbf{u})$ is given by (1.2). We investigate in this thesis the matrices for which the above condition is or is not satisfied.

In the radial case, it has been known since the work of Ball [8] that, for a large class of energy functions W, there exists a critical value λ_{cr} of the boundary displacement such that

(i) for $\lambda \leq \lambda_{cr}$ the unique radial energy minimiser is the homogeneous deformation $r(R) = \lambda R$;

(ii) for $\lambda > \lambda_{cr}$, the unique radial energy minimiser satisfies $r(0) > 0$, corresponding to a hole forming at the centre of the ball.

Upper and lower bounds on the value of λ_{cr} have been given by Stuart [50]. It is still a largely unresolved question whether the minimisers in the class of radial maps are still minimising if the class of competing deformations is enlarged to include non-radial maps. An important work in this direction is that of James and Spector [26], where it is shown that, for a special class of stored energy functions, the energy of a discontinuous radial deformation can be further reduced if one allows competing deformations producing thin filamentary voids in the body.

We conclude this outline of variational methods with a few remarks on the equilibrium equations of nonlinear elasticity. Although the original motivation for minimising the elastic energy (1.2) was to find solutions of (1.9), it is in fact a nontrivial and largely open question as to whether the minimisers obtained satisfy the weak form of (1.9). The difficulty is that, if $\mathbf{u}$ is a minimiser of E in a class of deformations in $W^{1,1}(\Omega;\mathbb{R}^3)$ satisfying (1.1), then to derive (1.9) one would usually consider variations $\mathbf{u}+t\varphi$, where $\varphi \in C_0^\infty(\Omega;\mathbb{R}^3)$, and it is not clear whether any such variation would have finite energy, since $\det(\nabla\mathbf{u}+t\nabla\varphi)$ may be negative on a set of positive measure. However, Ball [7] observed that it is still possible to show that such minimisers do satisfy weak forms of some alternative equilibrium equations, such as the energy-momentum equations

$$\frac{\partial}{\partial x^\alpha}\left[W(\nabla\mathbf{u}(\mathbf{x}))\delta_\alpha^\beta - \frac{\partial u^k}{\partial x^\beta}\frac{\partial W}{\partial F_\alpha^k}(\nabla\mathbf{u}(\mathbf{x}))\right] = 0 \quad \text{for } \beta = 1,2,3, \tag{1.28}$$

see [13, 6] for a complete proof. The two forms are equivalent for smooth invertible equilibria, but for singular or discontinuous equilibria weak solutions of (1.9) and (1.28) can be genuinely different.

1.2 Outline of the Thesis

Most of the results of the thesis concern the $W^{1,p}$-quasiconvexity over various classes of deformations of the stored energy functions

$$W_h(\mathbf{F}) = |\mathbf{F}|^p + h(\det\mathbf{F}) \quad \text{for all } \mathbf{F} \in M_+^{3\times 3}, \tag{1.29}$$

where $2 \leq p < 3$ and $h : (0, \infty) \to [0, \infty)$ is a C^1 function satisfying

$$h \text{ convex}, \; h(s) \to \infty \text{ as } s \to 0^+, \; \frac{h(s)}{s} \to \infty \text{ as } s \to \infty. \tag{1.30}$$

This stored energy function is on the one hand representative of the polyconvex stored energy functions encountered in elasticity and, on the other hand, has some special features which simplify the analysis. Many of our results extend to more general polyconvex stored energy functions.

In our approach to the study of $W^{1,p}$-quasiconvexity of W_h, we devote much attention to the study of the $W^{1,p}$-quasiconvexity of the related model energy function

$$W_\alpha(\mathbf{F}) = |\mathbf{F}|^p + \alpha \det \mathbf{F} \quad \text{for all } \mathbf{F} \in M_+^{3\times 3}. \tag{1.31}$$

The convexity of h yields a very simple connection between W_h given by (1.29) and W_α given by (1.31), by means of the inequality

$$\begin{aligned}
&\int_\Omega W_h(\nabla \mathbf{u})\, d\mathbf{x} - \int_\Omega W_h(\nabla \mathbf{u}_\mathbf{A}^{\text{hom}})\, d\mathbf{x} \\
&\quad \geq \int_\Omega |\nabla \mathbf{u}|^p - |\nabla \mathbf{u}_\mathbf{A}^{\text{hom}}|^p \, d\mathbf{x} + \int_\Omega h'(\det \mathbf{A})(\det \nabla \mathbf{u} - \det \nabla \mathbf{u}_\mathbf{A}^{\text{hom}})\, d\mathbf{x}. \\
&\quad = \int_\Omega W_\alpha(\nabla \mathbf{u})\, d\mathbf{x} - \int_\Omega W_\alpha(\nabla \mathbf{u}_\mathbf{A}^{\text{hom}})\, d\mathbf{x}, \qquad \text{where } \alpha := h'(\det \mathbf{A}).
\end{aligned} \tag{1.32}$$

Based upon this inequality, sufficient conditions for the $W^{1,p}$-quasiconvexity of W_α lead to sufficient conditions for the $W^{1,p}$-quasiconvexity of W_h.

Chapter 2 is devoted to the study of sufficient conditions for the quasiconvexity of W_h given by (1.29). An early result on this problem was that of Spector [48], who proved that, if $\mathbf{A}$ is such that

$$h'(\det \mathbf{A}) \leq 0,$$

then W_h is $W^{1,p}$-quasiconvex at $\mathbf{A}$ over the class of all mappings $\mathbf{u}$ with $\mathbf{u} - \mathbf{A}\mathbf{x} \in W_0^{1,p}(\Omega; \mathbb{R}^3)$, $\det \nabla \mathbf{u} > 0$ a.e., and for which

$$\int_\Omega [\det \mathbf{A} - \det \nabla \mathbf{u}]\, d\mathbf{x} \geq 0.$$

(In particular, in view of (1.53), this is the case for all mappings $\mathbf{u}$ in $\mathcal{A}_{\mathbf{A},p}$.) This result was improved by Müller, Spector and Sivaloganathan [35], who showed that there exists a constant $k > 0$ such that if

$$h'(\det \mathbf{A})|\mathbf{A}|^{3-p} \leq k, \tag{1.33}$$

then W_h is $W^{1,p}$-quasiconvex at $\mathbf{A}$ over the class $\mathcal{A}_{\mathbf{A},p}$. However, they did not give any explicit estimates on the value of k. Here we give an explicit estimate on k in Theorem 2.3. This is obtained from an explicit value, given in Theorem 2.1, of a constant μ in an inequality from [35] which bounds the integral of the difference of the Jacobians of two mappings, one of which is affine, in terms of the L^p norm of the difference of their gradients. Namely, for every n by n matrix $\mathbf{A}$ with positive determinant and for every bounded open region $\Omega \subset \mathbb{R}^n$,

$$\int_\Omega [\det \mathbf{A} - \det \nabla \mathbf{u}(\mathbf{x})]\, d\mathbf{x} \leq \mu |\mathbf{A}|^{n-p} \int_\Omega |\mathbf{A} - \nabla \mathbf{u}(\mathbf{x})|^p\, d\mathbf{x}, \tag{1.34}$$

for all $\mathbf{u} \in \mathcal{A}_{\mathbf{A},p}$. The value given here of the constant μ in (1.34) significantly improves the value which could be obtained by the arguments in [35]. The inequality (1.34) is in fact useful for the study of $W^{1,p}$-quasiconvexity for stored energy functions more general than (1.29), see [35] for details, and such explicit estimates are important for the determination of lower bounds on critical cavitation loads in elastic solids. As in [35], our approach consists in expressing the left-hand side of (1.34) in terms of the singular part of the distributional Jacobian of $\mathbf{u}$, estimating locally this singular measure by using the Isoperimetric Inequality, and finally using a covering argument.

We also derive, based on a different approach, an inequality which leads to a sufficient condition of the form

$$h'(\det \mathbf{A})(\det \mathbf{A})^{(3-p)/3} \leq \bar{k} \tag{1.35}$$

for the $W^{1,p}$-quasiconvexity of W_h given by (1.29) in the class of deformations producing a single hole anywhere in the material.

The main result of Chapter 3 is Theorem 3.1, which gives a necessary and sufficient condition for the $W^{1,p}$-quasiconvexity of the model energy function W_α given by (1.31) at $\lambda\mathbf{I}$ over the class of deformations opening a single hole anywhere in the material. We deal with the case $p \in [2, 3)$, and fully extend the result of

Sivaloganathan [41] for $p = 2$. The condition obtained in Theorem 3.1 takes the form

$$\alpha\lambda^{3-p} \leq \Upsilon_p, \tag{1.36}$$

where Υ_p is determined explicitly. This leads to the condition obtained in Theorem 3.4

$$\lambda^{3-p}h'(\lambda^3) \leq \Upsilon_p \tag{1.37}$$

as a sufficient condition for the $W^{1,p}$-quasiconvexity of W_h given by (1.29) over the same class of deformations. The condition (1.37) is shown in Theorem 3.12 to be optimal.

The proof of Theorem 3.1 is done in several steps. After some rescalings, it turns out that it suffices to work with mappings of the unit ball belonging to the class $\mathcal{A}_{\mathbf{I},p}(\mathbf{0})$ (see (1.26) for the definition of this class). The key step is to find a infimiser (in the sense that (1.38) holds) of the integral of the first term in (1.31) under the constraint that the integral of the second term is constant. Namely, for each $V \in (0, 4\pi/3]$, let us consider $\beta \in (0,1]$ such that $V = 4\pi\beta^3/3$. We show that there exists a radial mapping $\mathbf{u}_\beta(\mathbf{x}) = r_\beta(|\mathbf{x}|)\frac{\mathbf{x}}{|\mathbf{x}|}$, where r_β is a C^1 function on the interval $[0,1]$ which takes the value β on some interval $(0,\varepsilon)$ (where ε depends on β) and satisfies the radial Euler-Lagrange equation for the functional $\mathbf{F} \mapsto |\mathbf{F}|^p$ on the interval $[\varepsilon, 1]$. Moreover, $\mathbf{u}_\beta$ satisfies

$$\inf\left\{\int_\Omega |\nabla\mathbf{u}|^p\,d\mathbf{x} : \mathbf{u} \in \mathcal{A}_{\mathbf{I},p}(\mathbf{0}),\ m_{\mathbf{u}}(\overline{\Omega}) = V\right\} = \int_\Omega |\nabla\mathbf{u}_\beta|^p\,d\mathbf{x}. \tag{1.38}$$

Proving the existence of the function r_β is more complicated for $p \in (2,3)$ than for $p = 2$, where the corresponding radial Euler-Lagrange equations are linear and can be solved explicitly, the solutions being

$$r_\beta(R) = cR + \frac{d}{2R^2} \quad \text{for } R \in [\varepsilon, 1],$$

where c, d are constants. For general $p \in (2,3)$, the existence of r_β is obtained by a shooting argument. The proof of (1.38) makes essential use of an isoperimetric estimate. (We wish to emphasize here that the mappings in the class $\mathcal{A}_{\mathbf{I},p}(\mathbf{0})$ need not be radial.) Unlike the case $p = 2$, where the energy associated to W_α can be calculated explicitly for all the functions $\mathbf{u}_\beta$, and its minimiser with respect to

$\beta \in (0,1]$ can be determined, this is not possible for $p \in (2,3)$. Instead, we show that Υ_p can be calculated by means of the formula

$$\Upsilon_p = \lim_{\beta \searrow 0} \frac{I(r_\beta) - I(id)}{\beta^3/3}, \tag{1.39}$$

where $id : [0,1] \to [0,1]$ is given by $id(R) = R$ for all $R \in [0,1]$, and

$$I(r) := \int_0^1 R^2 \left[(r'(R))^2 + 2\left(\frac{r(R)}{R}\right)^2 \right]^{p/2} dR \qquad \text{for all } r \in W^{1,1}(0,1).$$

From (1.39), the value of Υ_p is calculated by making use of conservation laws satisfied by the radial equilibrium solutions r_β on the interval $[\varepsilon, 1]$.

Chapter 4 is mainly devoted to the study of necessary conditions for the $W^{1,p}$-quasiconvexity of W_h given by (1.29). We use an elementary approach based on comparing the energy of the homogeneous deformation with that of a one-parameter family of cavitating deformations, and we take a suitable limit with respect to the parameter. We are not aware of any other instance where this method has been used in the literature. By results in Section 1.4, the case of general affine boundary conditions $\mathbf{u}(\mathbf{x}) = \mathbf{A}\mathbf{x}$ on ∂B_1 reduces to that when $\mathbf{A}$ is a diagonal matrix. In this case, the test deformations we consider are of the type $\mathbf{u} = \mathbf{A}\mathbf{v}$, where $\mathbf{v}$ is radial, and the necessary condition obtained in Theorem 4.1 takes the form

$$(\det \mathbf{A}) h'(\det \mathbf{A}) \le \Lambda_p |\mathbf{A}|^p, \tag{1.40}$$

where Λ_p is explicitly determined. When $\mathbf{A} = \lambda \mathbf{I}$, the method can be slightly refined, and in this case our results recover some due to Stuart [50]. The necessary condition obtained in Theorem 4.3 takes the form

$$\lambda^{3-p} h'(\lambda^3) \le 3^{p/2} \Gamma_p, \tag{1.41}$$

where Γ_p is explicitly determined; the inequality $\Gamma_p \le \Lambda_p$ holds. When $p = 2$, the condition (1.41) is shown in Theorem 4.6 to be optimal for the $W^{1,2}$-quasiconvexity of W_h in the class of radial deformations.

Returning to the case of boundary conditions of the type $\mathbf{u}(\mathbf{x}) = \mathbf{A}\mathbf{x}$ on $\partial\Omega$, where $\mathbf{A}$ is a diagonal matrix, in Section 4.4 we propose a new class of test deformations, whose characteristic feature is that they map any sphere centred at $\mathbf{0}$ onto an ellipsoid centred at $\mathbf{0}$. This new class is successfully used to prove in Theorem 4.10 a necessary condition for the $W^{1,2}$-quasiconvexity of the functional W_α given by (1.31), a condition which coincides with the necessary and sufficient condition obtained for the same functional in Chapter 3 in the case $\mathbf{A} = \lambda\mathbf{I}$. One anticipates that this class of test deformations may find further applications in the study of quasiconvexity in nonlinear elasticity.

Chapter 5 is motivated by the problem of comparing the energy of mappings having singularities at different points in the unit ball. This is currently an important open problem for which there seems to be a lack of efficient methods. In [46, 47], linearization methods were used to compare the energy of such mappings satisfying $\mathbf{u}(\mathbf{x}) = \lambda\mathbf{x}$ on ∂B_1, where λ is bigger than and very close to λ_{cr} (the value of the boundary displacements at which cavitation becomes energy favourable), but those results depend on a number of assumptions whose validity in practice is not known. Here we take a different approach. We consider the stored energy given by (1.31) and relate the above mentioned problem to one of comparing the energy

$$E(\mathbf{u}) = \int_{B_1 \setminus B_\varepsilon(\mathbf{a})} |\nabla\mathbf{u}|^p \, d\mathbf{x} \tag{1.42}$$

of mappings $\mathbf{u}$ defined on spherical shells $B_1 \setminus B_\varepsilon(\mathbf{a})$, where $\mathbf{a}$ varies in B_1. For any number n of space dimensions and any $p \geq 2$, for any $\mathbf{a}$ such that $B_\varepsilon(\mathbf{a}) \subset B_1$, the energy given by (1.42) has a minimum in the class of mappings $\mathbf{u} \in W^{1,p}(B_1 \setminus B_\varepsilon(\mathbf{a}); \mathbb{R}^n)$ with $\mathbf{u}(\mathbf{x}) = \mathbf{x}$ on ∂B_1. We conjecture that, as $\mathbf{a}$ varies in $B_1 \setminus B_{1-\varepsilon}$, the above minimum of the energy takes its smallest value when $\mathbf{a} = \mathbf{0}$. Although the situation that motivated this conjecture is that when $n = 3$, $p \in [2, 3)$, we are able to prove the conjecture only in the case $p = 2$, $n = 2$. Unfortunately, the current method relies heavily on conformal mappings, and cannot obviously be extended to higher dimensions or exponents $p \neq 2$.

In Chapter 6 we consider the uniqueness of solutions of the equilibrium equations of nonlinear elasticity in star-shaped domains. We relax the regularity assumptions of Knops and Stuart [28], who proved that if the stored energy function is rank-one convex on $M_+^{3\times 3}$ and strictly quasiconvex at $\mathbf{A}$, then the only solution $\mathbf{u} \in C^2(\Omega; \mathbb{R}^3) \cap C^1(\overline{\Omega}; \mathbb{R}^3)$ of the Euler-Lagrange equations under affine boundary conditions $\mathbf{u}(\mathbf{x}) = \mathbf{A}\mathbf{x}$ on $\partial\Omega$ is the homogeneous deformation. Our approach is related to that of Taheri [51], who considered simultaneously weak solutions (which are smooth near $\partial\Omega$) of the Euler-Lagrange equations (1.9) and the energy-momentum equations (1.28). In both these works the uniqueness is derived from a representation formula for the energy of a critical point as a boundary integral

$$3E(\mathbf{u}) = \int_{\partial\Omega} W(\nabla\mathbf{u})(\mathbf{x}\cdot N(\mathbf{x})) + \frac{\partial W}{\partial F}(\nabla\mathbf{u}) : [(\mathbf{u}(\mathbf{x}) - \nabla\mathbf{u}(\mathbf{x})\mathbf{x}) \otimes N(\mathbf{x})]\, d\mathcal{H}^2(\mathbf{x}),$$

where $N(\mathbf{x})$ is the outward unit normal at the point $\mathbf{x} \in \partial\Omega$. Here we prove in Theorem 6.3 that if the weak form of the Green divergence identity (which is well known to be valid for classical solutions of the Euler-Lagrange equations)

$$\frac{\partial}{\partial x^\alpha}\left[x^\alpha W(\nabla\mathbf{u}) + \left(u^i - \frac{\partial u^i}{\partial x^k}x^k\right)\frac{\partial W}{\partial F^i_\alpha}(\nabla\mathbf{u})\right] = 3W(\nabla\mathbf{u}),$$

is satisfied, then the above representation formula of the energy holds, which implies the uniqueness. We also show in Theorem 6.7 that the weak form of this identity is satisfied by local extrema (which are smooth near $\partial\Omega$) of the functional.

The remaining part of Chapter 1 contains background material and some consequences of $W^{1,p}$-quasiconvexity.

1.3 Background Material

Among the topics discussed in this section are some fundamental aspects of degree theory, a few notions and results from geometric measure theory, a special notion of invertibility of mappings, and a review of results on radial cavitation.

1.3.1 Vector and matrix notation

We use the following notation for vectors:

$\mathbf{a}^T = (a_1\, a_2 \ldots a_n)$ transpose of the vector $\mathbf{a}$.
$\mathbf{a}\cdot\mathbf{b} = \mathbf{a}^T\mathbf{b}$ Euclidean inner-product in $\mathbb{R}^n$.
$\mathbf{a}\otimes\mathbf{b} = \mathbf{a}\mathbf{b}^T = (a_i b_j)$ tensor product in $\mathbb{R}^n$.

We use the following notation for matrices:

$M^{n\times n}$ the set of square matrices of dimension n.
$M_{+}^{n\times n}$ the set of square matrices with positive determinant.
$\mathbf{A}^T$ transpose of the matrix $\mathbf{A}$.
$\mathbf{A}^{1/2}$ square root of a symmetric positive definite matrix $\mathbf{A}$.
$\mathbf{I} = (\delta_{ij})$ unit matrix.
$\mathrm{diag}(\mu_1, \mu_2, ..., \mu_n)$ diagonal matrix whose diagonal elements are $\mu_1, \mu_2, ..., \mu_n$ (in this order).
$\mathrm{tr}\,\mathbf{A}$ trace of the matrix $\mathbf{A}$.
$\det \mathbf{A}$ determinant of the matrix $\mathbf{A}$.
$\mathrm{adj}\,\mathbf{A}$ adjugate matrix of the matrix $\mathbf{A}$.
$\mathbf{A} : \mathbf{B} = \mathrm{tr}\,\mathbf{A}^T\mathbf{B}$ matrix inner product on $M^{n\times n}$.
$\|\mathbf{A}\| = \{\mathbf{A} : \mathbf{A}\}^{1/2}$ matrix norm on $M^{n\times n}$.
$O(n)$ the orthogonal group on $\mathbb{R}^n$.
$SO(n)$ the special orthogonal group on $\mathbb{R}^n$.

1.3.2 Degree Theory

In the following, we assume that D is a bounded open subset of $\mathbb{R}^n$. The purpose of this subsection is to define the topological degree for continuous functions on the boundary of D and present some of its properties. We refer for more details to the books by Fonseca and Gangbo [18], Lloyd [30] and Schwartz [39].

We start by introducing the definition of the degree for C^1 functions.

Definition 1.1. *Let $\mathbf{f} \in C^1(\overline{D}; \mathbb{R}^n)$ and let $\mathbf{x}_0 \in \overline{D}$. We say that $\mathbf{x}_0$ is a critical point of $\mathbf{f}$ if $\det \nabla \mathbf{f}(\mathbf{x}_0) = 0$. For such $\mathbf{x}_0$, $\mathbf{f}(\mathbf{x}_0)$ is called a critical value of $\mathbf{f}$. We define*

$$Z_{\mathbf{f}} = \left\{\mathbf{x}_0 \in \overline{D} : \det \nabla \mathbf{f}(\mathbf{x}_0) = 0\right\},$$

the set of critical points of $\mathbf{f}$, and $\mathbf{f}(Z_{\mathbf{f}})$ is called the crease of $\mathbf{f}$.

We next present Sard's Lemma, asserting that the crease of a C^1 function is small.

Lemma 1.2. *Let $\mathbf{f} \in C^1(\overline{D}; \mathbb{R}^n)$. Then $\mathbf{f}(Z_{\mathbf{f}})$ is a set of measure zero in $\mathbb{R}^n$.*

Next note that the inverse image of a crease point may be an infinite set, while for all the other points in $\mathbb{R}^n \setminus \mathbf{f}(\partial D)$ the inverse image is a finite set as shown by the following lemma.

Lemma 1.3. *Let* $\mathbf{f} \in C^1(\overline{D}; \mathbb{R}^n)$ *and suppose that* $\mathbf{y} \notin \mathbf{f}(\partial D) \cup \mathbf{f}(Z_\mathbf{f})$. *Then its inverse image* $\mathbf{f}^{-1}\{\mathbf{y}\}$ *is a finite set (possibly empty).*

We can now define the degree of $\mathbf{f}$ at $\mathbf{y}$ when $\mathbf{f}$ is a C^1 function and $\mathbf{y}$ is not a critical value of $\mathbf{f}$.

Definition 1.4. *If* $\mathbf{y} \notin \mathbf{f}(\partial D) \cup \mathbf{f}(Z_\mathbf{f})$, *we define the* **degree of f at y with respect to** D, *by*

$$\deg(\mathbf{f}, D, \mathbf{y}) := \sum_{\mathbf{x} \in \mathbf{f}^{-1}\{\mathbf{y}\}} \operatorname{sgn}[\det \nabla \mathbf{f}(\mathbf{x})],$$

where $\operatorname{sgn} t := t/|t|, t \in \mathbb{R} \setminus \{0\}$.

Remark 1.5. Note that since $\mathbf{y}$ is not in the crease of $\mathbf{f}$, the summation in the above formula is finite and hence the degree is well defined.

The task ahead is to remove the restrictions $\mathbf{f} \in C^1$ and $\mathbf{y} \notin \mathbf{f}(Z_\mathbf{f})$ imposed in the Definition 1.4. This is done by a process of approximation.

The following theorem asserts that if $\mathbf{f} \in C^1(\overline{D}; \mathbb{R}^n)$, $\mathbf{y} \notin \mathbf{f}(\partial D) \cup \mathbf{f}(Z_\mathbf{f})$, and $\mathbf{g}$ is sufficiently near $\mathbf{f}$ in the C^1 topology, then $\deg(\mathbf{g}, D, \mathbf{y})$ is defined and is equal to $\deg(\mathbf{f}, D, \mathbf{y})$.

Theorem 1.6. *Suppose that* $\mathbf{f} \in C^1(\overline{D}; \mathbb{R}^n)$ *and* $\mathbf{y} \notin \mathbf{f}(\partial D) \cup \mathbf{f}(Z_\mathbf{f})$. *Then there exists* $\delta > 0$, *depending on* $\mathbf{y}$ *and* $\mathbf{f}$, *such that, if* $\|\mathbf{f} - \mathbf{g}\|_1 < \delta$, *then* $\mathbf{y} \notin \mathbf{g}(\partial D) \cup \mathbf{g}(Z_\mathbf{g})$ *and* $\deg(\mathbf{f}, D, \mathbf{y}) = \deg(\mathbf{g}, D, \mathbf{y})$.

We next express the degree of $\mathbf{f}$ as an integral involving an averaging kernel.

Theorem 1.7. *Suppose that* $\mathbf{f} \in C^1(\overline{D}; \mathbb{R}^n)$ *and* $\mathbf{y} \notin \mathbf{f}(\partial D) \cup \mathbf{f}(Z_\mathbf{f})$. *Let* $\theta_\varepsilon : \mathbb{R}^n \to \mathbb{R}$ *be a continuous function such that*

$$K_\varepsilon \equiv \operatorname{supp} \theta_\varepsilon \subset B(\mathbf{0}, \varepsilon), \quad \int_{\mathbb{R}^n} \theta_\varepsilon(\mathbf{x})\, d\mathbf{x} = 1.$$

Then there exists ε_0 (depending on $\mathbf{y}$ and $\mathbf{f}$) such that, if $0 < \varepsilon < \varepsilon_0$, then

$$\deg(\mathbf{f}, D, \mathbf{y}) = \int_D \theta_\varepsilon(\mathbf{f}(\mathbf{x}) - \mathbf{y}) \det \nabla\mathbf{f}(\mathbf{x})\, d\mathbf{x}. \tag{1.43}$$

The next stage is to examine the effect on $\deg(\mathbf{f}, D, \mathbf{y})$ of changes in $\mathbf{y}$.

Theorem 1.8. *Let $\mathbf{f} \in C^1(\overline{D}; \mathbb{R}^n)$. Suppose that $\mathbf{y_1}$ and $\mathbf{y_2}$ are in the same component of $\mathbb{R}^n \setminus \mathbf{f}(\partial D)$ and that neither is in the crease of $\mathbf{f}$. Then*

$$\deg(\mathbf{f}, D, \mathbf{y_1}) = \deg(\mathbf{f}, D, \mathbf{y_2}).$$

We can now remove the restriction $\mathbf{y} \notin \mathbf{f}(Z_\mathbf{f})$ in our definition of degree.

Definition 1.9. *If $\mathbf{f} \in C^1(\overline{D}; \mathbb{R}^n)$ and $\mathbf{y} \notin \mathbf{f}(\partial D)$ but $\mathbf{y} \in \mathbf{f}(Z_\mathbf{f})$, we define* $\deg(\mathbf{f}, D, \mathbf{y})$ *to be* $\deg(\mathbf{f}, D, \mathbf{q})$*, where $\mathbf{q}$ is any point such that $\mathbf{q} \notin \mathbf{f}(Z_\mathbf{f})$ and* $|\mathbf{q} - \mathbf{y}| < \operatorname{dist}(\mathbf{y}, \mathbf{f}(\partial D))$.

We now state some properties of $\deg(\mathbf{f}, D, \mathbf{y})$ for C^1 functions, regardless of whether $\mathbf{y}$ is a crease point or not. These properties will be used in the final stage of our definition.

Definition 1.10. *A C^1 homotopy between elements $\mathbf{f}$ and $\mathbf{g}$ of $C^1(\overline{D}; \mathbb{R}^n)$ is a function $H : \overline{D} \times [0, 1] \to \mathbb{R}^n$ such that, if H_t denotes the function $\mathbf{x} \mapsto H(\mathbf{x}, t)$, then*

(i) $H_t \in C^1(\overline{D}; \mathbb{R}^n), \quad 0 \le t \le 1$,

(ii) $H_0(\mathbf{x}) = \mathbf{f}(\mathbf{x}), \quad H_1(\mathbf{x}) = \mathbf{g}(\mathbf{x})$, *for every $\mathbf{x} \in \overline{D}$,*

(iii) $\lim_{s \to t} \|H_t - H_s\|_1 = 0$.

Theorem 1.11. *Let $\mathbf{f} \in C^1(\overline{D}; \mathbb{R}^n)$.*

(i) $\deg(\mathbf{f}, D, \mathbf{y})$ *is constant on connected components of $\mathbb{R}^n \setminus \mathbf{f}(\partial D)$.*

(ii) *If $\mathbf{y} \notin \mathbf{f}(\partial D)$, then there exists ε, depending on $\mathbf{y}$ and $\mathbf{f}$, such that* $\deg(\mathbf{f}, D, \mathbf{y}) = \deg(\mathbf{g}, D, \mathbf{y})$ *whenever $\|\mathbf{f} - \mathbf{g}\|_1 < \varepsilon$.*

(iii) *Let $H(\mathbf{x}, t)$ be a C^1 homotopy between $\mathbf{f}$ and $\mathbf{g}$; if $\mathbf{y} \notin H(\partial D, t)$ for all $t \in [0, 1]$, then* $\deg(\mathbf{f}, D, \mathbf{y}) = \deg(\mathbf{g}, D, \mathbf{y})$.

An immediate consequence of Theorem 1.11 is that the topological degree depends only on the boundary values of the function in question.

Corollary 1.12. *Let* $\mathbf{f} \in C^1(\overline{D};\mathbb{R}^n)$ *be such that* $\mathbf{f}(\mathbf{x}) = \mathbf{g}(\mathbf{x})$ *for* $\mathbf{x} \in \partial D$. *Then for every* $\mathbf{y} \in \mathbb{R}^n \setminus \mathbf{f}(\partial D)$,

$$\deg(\mathbf{f}, D, \mathbf{y}) = \deg(\mathbf{g}, D, \mathbf{y}).$$

We now come to the final stage of the definition of the degree. We suppose only that $\mathbf{f} \in C(\overline{D};\mathbb{R}^n)$. The degree of $\mathbf{f}$ is then the degree of a sufficiently good C^1 approximation to $\mathbf{f}$. That a definition of degree is possible for non-differentiable functions demonstrates the topological nature of the concept; in these terms the analytic formulation we have pursued is merely a means of calculation.

Definition 1.13. *Suppose that* $\mathbf{f} \in C(\overline{D};\mathbb{R}^n)$ *and* $\mathbf{y} \notin \mathbf{f}(\partial D)$. *Define* $\deg(\mathbf{f}, D, \mathbf{y})$ *to be* $\deg(\mathbf{g}, D, \mathbf{y})$, *where* $\mathbf{g}$ *is any function in* $C^1(\overline{D};\mathbb{R}^n)$ *satisfying*

$$|\mathbf{f}(\mathbf{x}) - \mathbf{g}(\mathbf{x})| < \operatorname{dist}(\mathbf{y}, \mathbf{f}(\partial D))$$

for $\mathbf{x} \in \overline{D}$.

Remark 1.14. The definition of the degree is independent of the choice of the C^1 function $\mathbf{g}$, which can be chosen such that $\mathbf{y} \notin \mathbf{g}(Z_{\mathbf{g}})$.

Definition 1.15. *A homotopy between elements* $\mathbf{f}$ *and* $\mathbf{g}$ *of* $C(\overline{D};\mathbb{R}^n)$ *is a continuous function* $H : \overline{D} \times [0,1] \to \mathbb{R}^n$ *such that, if* H_t *denotes the function* $\mathbf{x} \mapsto H(\mathbf{x}, t)$, *then* $H_0(\mathbf{x}) = \mathbf{f}(\mathbf{x})$ *and* $H_1(\mathbf{x}) = \mathbf{g}(\mathbf{x})$ *for all* $\mathbf{x} \in \overline{D}$.

Theorem 1.16. *Let* $\mathbf{f} \in C(\overline{D};\mathbb{R}^n)$.

(i) $\deg(\mathbf{f}, D, \mathbf{y})$ *is constant on connected components of* $\mathbb{R}^n \setminus \mathbf{f}(\partial D)$; *its value is* 0 *in the unique unbounded component of* $\mathbb{R}^n \setminus \mathbf{f}(\partial D)$.

(ii) *Let* $\mathbf{y} \notin \mathbf{f}(\partial D)$. *If* $\mathbf{g} \in C(\overline{D};\mathbb{R}^n)$ *is such that* $|\mathbf{f}(\mathbf{x}) - \mathbf{g}(\mathbf{x})| < \operatorname{dist}(\mathbf{y}, \mathbf{f}(\partial D))$ *for* $\mathbf{x} \in \overline{D}$, *then* $\deg(\mathbf{g}, D, \mathbf{y})$ *is well defined and* $\deg(\mathbf{f}, D, \mathbf{y}) = \deg(\mathbf{g}, D, \mathbf{y})$.

(iii) *Let* $H(\mathbf{x}, t)$ *be a homotopy between* $\mathbf{f}$ *and* $\mathbf{g}$; *if* $\mathbf{y} \notin H(\partial D, t)$ *for all* $t \in [0,1]$, *then* $\deg(\mathbf{f}, D, \mathbf{y}) = \deg(\mathbf{g}, D, \mathbf{y})$.

Next we come to a simple but important fact, namely that $\deg(\mathbf{f}, D, \mathbf{y})$ depends only on the values $\mathbf{f}$ takes on ∂D.

Theorem 1.17. *Let* $\mathbf{f}, \mathbf{g} \in C(\overline{D}; \mathbb{R}^n)$ *be such that* $\mathbf{f} = \mathbf{g}$ *on* ∂D *and let* $\mathbf{y} \notin \mathbf{f}(\partial D)$. *Then* $\deg(\mathbf{f}, D, \mathbf{y}) = \deg(\mathbf{g}, D, \mathbf{y})$.

Suppose now that $\mathbf{f} \in C(\partial D; \mathbb{R}^n)$. By Tietze Extension Theorem, there exists a function $\tilde{\mathbf{f}} \in C(\overline{D}; \mathbb{R}^n)$ such that $\tilde{\mathbf{f}}|_{\partial D} = \mathbf{f}$.

Definition 1.18. *Let* $\mathbf{f} \in C(\partial D; \mathbb{R}^n)$ *and let* $\mathbf{y} \notin \mathbf{f}(\partial D)$. *Let* $\tilde{\mathbf{f}}$ *be a continuous extension of* $\mathbf{f}$ *to* $\overline{D}$. *We define* $\deg(\mathbf{f}, D, \mathbf{y})$ *for all points* $\mathbf{y} \notin \mathbf{f}(\partial D)$, *by*

$$\deg(\mathbf{f}, D, \mathbf{y}) := \deg(\tilde{\mathbf{f}}, D, \mathbf{y}).$$

Remark 1.19. Theorem 1.17 shows that the degree of $\mathbf{f}$ is independent of the choice of the extension $\tilde{\mathbf{f}}$ used.

Remark 1.20. Since in Definition 1.18 the degree was defined for continuous functions $\mathbf{f}$ on ∂D, it is more natural to use the notation $\deg(\mathbf{f}, \partial D, \mathbf{y})$ instead of $\deg(\mathbf{f}, D, \mathbf{y})$.

1.3.3 A Few Results from Geometric Measure Theory

Throughout the thesis, n-dimensional Lebesgue measure will be denoted by $\mathcal{L}^n$ and k-dimensional Hausdorff measure by $\mathcal{H}^k$. We write

$$B(\mathbf{a}, r) := \{\mathbf{y} \in \mathbb{R}^n : |\mathbf{y} - \mathbf{a}| < r\},$$

for the ball of radius r centred at $\mathbf{a} \in \mathbb{R}^n$.

Theorem 1.21. (Besicovitch Covering Theorem, [17, p. 30]). *There exists a constant* ϖ_n, *depending only on* n, *with the following property: if* $\mathcal{F}$ *is any collection of non-degenerate closed balls in* $\mathbb{R}^n$ *with*

$$\sup \left\{ \operatorname{diam} B \middle| B \in \mathcal{F} \right\} < \infty$$

and if A is the set of centers of balls in $\mathcal{F}$, then there exist $\mathcal{G}_1, \dots, \mathcal{G}_{\varpi_n} \subset \mathcal{F}$ such that each $\mathcal{G}_i$ $(i = 1, \dots, \varpi_n)$ is a countable collection of disjoint balls in $\mathcal{F}$ and

$$A \subset \bigcup_{i=1}^{\varpi_n} \bigcup_{B \in \mathcal{G}_i} B. \tag{1.44}$$

We now introduce the notion of a set of finite perimeter, and define the reduced boundary of such sets.

Definition 1.22. ([17, p. 167]). *Let $A \subset \mathbb{R}^n$ be a $\mathcal{L}^n$-measurable set. We say that A has* **finite perimeter** *if*

$$\sup \left\{ \int_A \operatorname{div} \varphi \, d\mathcal{L}^n : \varphi \in C_0^1(\mathbb{R}^n; \mathbb{R}^n), |\varphi| \leq 1 \right\} < \infty.$$

Theorem 1.23. ([17, p. 167-169]). *Let $A \subset \mathbb{R}^n$ be a set of finite perimeter. Then there exists a Radon measure $||\partial A||$ on $\mathbb{R}^n$ and a $||\partial A||$-measurable function $\nu_A : \mathbb{R}^n \to \mathbb{R}^n$ such that*

(i) $|\nu_A(\mathbf{x})| = 1$, *$||\partial A||$-almost everywhere,*

(ii)
$$\int_A \operatorname{div} \varphi \, d\mathcal{L}^n = \int_{\mathbb{R}^n} \varphi \cdot \nu_A \, d||\partial A|| \quad \textit{for all } \varphi \in C_0^1(\mathbb{R}^n; \mathbb{R}^n).$$

For any set $D \subset \mathbb{R}^n$, any Radon measure μ on R^n and any μ-measurable function f, we denote by $\fint_D f \, d\mu$ the average value with respect to μ of f over D.

Definition 1.24. ([17, p. 194]). *Let $\mathbf{x} \in \mathbb{R}^n$. We say that $\mathbf{x} \in \partial^* A$, the* **reduced boundary** *of A, if*

(i) $||\partial A||(B(\mathbf{x}, r)) > 0$ *for all* $r > 0$,

(ii)
$$\lim_{r \to 0} \fint_{B(\mathbf{x},r)} \nu_A \, d||\partial A|| = \nu_A(\mathbf{x}),$$

(iii) $|\nu_A(\mathbf{x})| = 1$.

Examples.([17, p. 171],[20, p. 44]).

(i) Let A be an open set in $\mathbb{R}^n$ whose boundary is a C^1 hypersurface. Then $||\partial A||$ is the restriction of $\mathcal{H}^{n-1}$ to ∂A (the boundary of A), ν_A is the unit outer normal vector to ∂A, and $\partial^* A = \partial A$.

(ii) Let A be the unit square in $\mathbb{R}^2$. Then $||\partial A||$ is the restriction of $\mathcal{H}^1$ to ∂A, ν_A is the unit outer normal vector to ∂A, which is well defined except at the corners, and $\partial^* A$ is the boundary of A without the corners.

Proposition 1.25. (Isoperimetric Inequality, [17, p. 190, p. 205].) *For $n \geq 2$, let $\omega := n^{-1}\mathcal{L}^n(B(0,1))^{-1/n}$. Then*

$$\mathcal{L}^n(A)^{(n-1)/n} \leq \omega \mathcal{H}^{n-1}(\partial^* A) \tag{1.45}$$

for every bounded measurable set $A \in \mathbb{R}^n$ of finite perimeter, where $\partial^* A$ denotes the reduced boundary of A.

1.3.4 Sobolev Spaces

In the following, D will denote a non-empty, open subset of $\mathbb{R}^n, n \geq 2$. By $L^p(D)$ and $W^{1,p}(D)$ we denote the usual spaces of p-summable and Sobolev functions, respectively. We use the notations $L^p(D;\mathbb{R}^m)$ and $W^{1,p}(D;\mathbb{R}^m)$ for vector-valued maps. A function is in $L^p_{\text{loc}}(D)$ if it is in $L^p(E)$ for all $E \subset\subset D$. Sobolev spaces on manifolds are defined by the use of local charts, see [23, 32]. We do not identify functions that agree almost everywhere. We use the shorthand notation $\mathbf{u} \in W^{1,p}(D;\mathbb{R}^m)$ to indicate that $\mathbf{u}$ is a representative of an equivalence class that is contained in $W^{1,p}(D;\mathbb{R}^m)$.

Since we are interested in pointwise properties of Sobolev functions, as well as their restrictions of to lower dimensional sets, it is useful to consider a particular representative.

Definition 1.26. *Let $\mathbf{u} \in W^{1,p}(D;\mathbb{R}^n)$. We define the* **precise representative** $\mathbf{u}^* : D \to \mathbb{R}^n$ *by*

$$\mathbf{u}^*(\mathbf{x}) = \begin{cases} \lim\limits_{r\to 0^+} \fint_{B(\mathbf{x},r)} \mathbf{u}(\mathbf{z})\, d\mathbf{z}, & \textit{if the limit exists,} \\ 0, & \textit{otherwise,} \end{cases}$$

where $\fint_A$ denotes the average value of the integrand over A.

The precise representative satisfies many important properties, some of which are summarised in the following result. For $\mathbf{a} \in D$ we let

$$r_{\mathbf{a}} := \operatorname{dist}(\mathbf{a}, \partial D),$$

which is the distance from $\mathbf{a}$ to the boundary of D.

Proposition 1.27. ([34, p. 14-15], [23, Theorem 2.8]).

(i) *Let $p \in [1, \infty]$. For every $\mathbf{u} \in W^{1,p}(D; \mathbb{R}^n)$ and for every $\mathbf{a} \in D$ there exists an $\mathcal{L}^1$ null set $N_{\mathbf{a}}(\mathbf{u})$ such that, for all $r \in (0, r_{\mathbf{a}}) \setminus N_{\mathbf{a}}(\mathbf{u})$, $\mathbf{u}^*|_{\partial B(\mathbf{a},r)} \in W^{1,p}(\partial B(\mathbf{a}, r); \mathbb{R}^n)$.*

(ii) (Sobolev Imbedding Theorem) *Suppose that $p > n - 1$. Then there exists a constant $C > 0$ such that, for all $\mathbf{u} \in W^{1,p}(D; \mathbb{R}^n)$, for all $\mathbf{a} \in D$ and for all $r \in (0, r_{\mathbf{a}}) \setminus N_{\mathbf{a}}(\mathbf{u})$,*

$$\sup_{\mathbf{x},\mathbf{y} \in \partial B(\mathbf{a},r)} \frac{|\mathbf{u}^*(\mathbf{x}) - \mathbf{u}^*(\mathbf{y})|}{|\mathbf{x} - \mathbf{y}|^{\alpha}} \leq C \left(\int_{\partial B(\mathbf{a},r)} |\nabla \mathbf{u}|^p \, d\mathcal{H}^{n-1} \right)^{1/p},$$

where $\alpha := 1 - (n-1)/p$.

In particular, $\mathbf{u}^*|_{\partial B(\mathbf{a},r)} \in C^0(\partial B(\mathbf{a}, r); \mathbb{R}^n)$ for all $r \in (0, r_{\mathbf{a}}) \setminus N_{\mathbf{a}}(\mathbf{u})$.

Proposition 1.28. ([34, Proposition 2.7]). *Let Γ be an oriented, smooth, $(n-1)$-dimensional manifold. Suppose that $\mathbf{u} \in W^{1,p}(\Gamma; \mathbb{R}^n) \cap C^0(\Gamma; \mathbb{R}^n)$, with $p > n-1$. Then for any $\mathcal{H}^{n-1}$ measurable set $A \subset \Gamma$,*

$$\mathcal{H}^{n-1}(\mathbf{u}(A)) \leq (n-1)^{(1-n)/2} \int_A |\nabla \mathbf{u}|^{n-1} \, d\mathcal{H}^{n-1}. \tag{1.46}$$

(We denote by $\nabla \mathbf{u}$ the tangential derivative of $\mathbf{u}$.)

1.3.5 The Invertibility Condition (INV)

In this subsection we define the invertibility condition (INV) and present some properties of mappings satisfying this condition. Our exposition is based on the paper of Müller and Spector [34], to which we refer for proofs and further details.

In nonlinear elasticity one is interested in globally invertible maps since, in general, matter cannot interpenetrate itself.

Definition 1.29. *We say that* $\mathbf{u} \in W^{1,1}(D;\mathbb{R}^n)$ *is* **invertible almost everywhere** *(or equivalently,* **one-to-one almost everywhere***) if there is a Lebesgue null set* $N \subset D$ *such that* $\mathbf{u}|_{D\setminus N}$ *is injective.*

We note that invertibility almost everywhere is a property of the equivalence class and not merely of the representative.

Definition 1.30. *Let* $\mathbf{u} : \partial B(\mathbf{a}, r) \to \mathbb{R}^n$ *be a continuous function. We define the* **topological image** *of* $B(\mathbf{a}, r)$ *under* $\mathbf{u}$ *by*

$$\operatorname{im}_T(\mathbf{u}, B(\mathbf{a}, r)) := \{\mathbf{y} \in \mathbb{R}^n \setminus \mathbf{u}(\partial B(\mathbf{a}, r)) : \deg(\mathbf{u}, \partial B(\mathbf{a}, r), \mathbf{y}) \neq 0\}.$$

Thus the topological image of a ball $B(\mathbf{a}, r)$ under $\mathbf{u}$ is the topological image of the ball under any continuous function in $\overline{B(\mathbf{a}, r)}$ whose restriction to $\partial B(\mathbf{a}, r)$ is $\mathbf{u}$.

Definition 1.31. *We say that* $\mathbf{u} : D \to \mathbb{R}^n$ **satisfies condition** (INV) *provided that for every* $\mathbf{a} \in D$ *there exists an* $\mathcal{L}^1$ *null set* $N_{\mathbf{a}}$ *such that, for all* $r \in (0, r_{\mathbf{a}}) \setminus N_{\mathbf{a}}$, $\mathbf{u}|_{\partial B(\mathbf{a},r)}$ *is continuous,*

(i) $\mathbf{u}(\mathbf{x}) \in \operatorname{im}_T(\mathbf{u}, B(\mathbf{a}, r)) \cup \mathbf{u}(\partial B(\mathbf{a}, r))$ *for* $\mathcal{L}^n$ *a.e.* $\mathbf{x} \in \overline{B(\mathbf{a}, r)}$, *and*

(ii) $\mathbf{u}(\mathbf{x}) \in \mathbb{R}^n \setminus \operatorname{im}_T(\mathbf{u}, B(\mathbf{a}, r))$ *for* $\mathcal{L}^n$ *a.e.* $\mathbf{x} \in D \setminus \overline{B(\mathbf{a}, r)}$.

Remark 1.32. Fix $\mathbf{a} \in D$. Then (i) and (ii) can be thought of as the requirement that (almost) every shell centred at $\mathbf{a}$ is a solid, impenetrable two-dimensional body that is subject to a continuous deformation. Thus, all matter that was originally inside such a shell must remain inside, and all matter that was originally outside such a shell must remain outside, see Figure 1-1.

Proposition 1.33. ([34, Lemma 3.4]). *Let* $\mathbf{u} \in W^{1,p}_{loc}(D;\mathbb{R}^n)$ *with* $p > n-1$. *Suppose that* $\mathbf{u}^*$ *satisfies condition* (INV) *and that* $\det \nabla \mathbf{u} \neq 0$ *a.e.. Then* $\mathbf{u}$ *is one-to-one almost everywhere.*

Proposition 1.34. ([34, Remark 1, p. 17]). *Every mapping* $\mathbf{u} \in W^{1,n}_{loc}(D;\mathbb{R}^n)$ *satisfying condition* (INV) *is continuous.*

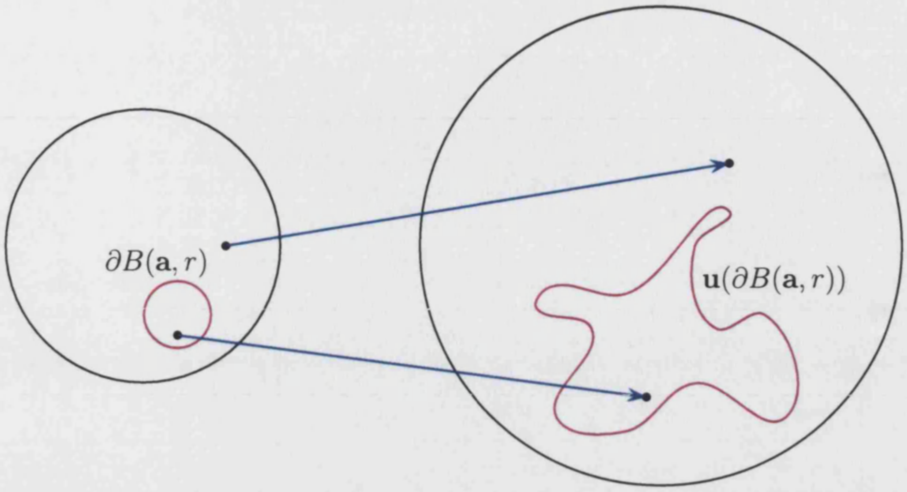

Figure 1-1: The (INV) condition

The next result gives further information on the values of the degree for certain Sobolev functions satisfying the (INV) condition.

Proposition 1.35. ([34, Lemma 3.5]). *Let* $\mathbf{u} \in W^{1,p}(D; \mathbb{R}^n)$ *with* $p > n-1$. *Assume that* $\mathbf{u}^*$ *satisfies condition* (INV) *and that* $\det \nabla \mathbf{u} \neq 0$ *a.e.. Fix* $\mathbf{a} \in D$.

(i) *Then there exists an* $\mathcal{L}^1$ *null set* $N_{\mathbf{a}}$ *such that for every* $r \in (0, r_{\mathbf{a}}) \setminus N_{\mathbf{a}}$,

$$\deg(\mathbf{u}, \partial B(\mathbf{a}, r), \mathbf{y}) \in \{-1, 0, 1\} \quad \text{for all} \quad \mathbf{y} \in \mathbb{R}^n \setminus \mathbf{u}(\partial B(\mathbf{a}, r)).$$

(ii) *If, in addition,* $\det \nabla \mathbf{u} > 0$ *a.e., then*

$$\deg(\mathbf{u}, \partial B(\mathbf{a}, r), \mathbf{y}) \in \{0, 1\} \quad \text{for all} \quad \mathbf{y} \in \mathbb{R}^n \setminus \mathbf{u}(\partial B(\mathbf{a}, r)). \tag{1.47}$$

(iii) *Conversely, if there is an* $r_0 \in (0, r_{\mathbf{a}}) \setminus N_{\mathbf{a}}$ *such that* (1.47) *holds, then*

$$\det \nabla \mathbf{u} > 0 \quad \text{a.e. in} \quad B(\mathbf{a}, r_0).$$

Proposition 1.36. ([34, Proof of Lemma 3.5]). *Let* $\mathbf{u} \in W^{1,p}_{loc}(D; \mathbb{R}^n)$ *with* $p > n-1$. *Assume that* $\mathbf{u}^*$ *satisfies condition* (INV) *and that* $\det \nabla \mathbf{u} \neq 0$ *a.e.. Then* *for every* $\mathbf{a} \in D$ *and almost every* $r \in (0, r_{\mathbf{a}})$ *the set* $\mathrm{im}_T(\mathbf{u}, B(\mathbf{a}, r))$ *has finite perimeter. Moreover, for such* r*, the reduced boundary satisfies*

$$\partial^* \mathrm{im}_T(\mathbf{u}, B(\mathbf{a}, r)) \subset \mathbf{u}(\partial B(\mathbf{a}, r)).$$

Proposition 1.37. ([34, Lemma 7.3]). *Let* $\mathbf{u} \in W^{1,p}(D; \mathbb{R}^n)$*, with* $p > n - 1$*. Suppose that* $\det \nabla \mathbf{u} \neq 0$ *a.e. and that* $\mathbf{u}^*$ *satisfies condition* (INV)*. Let* $\mathbf{a} \in D$*. Then there exists an* $\mathcal{L}^1$ *null set* $N_{\mathbf{a}}$ *such that the restriction of* $\mathbf{u}^*$ *to* $\partial B(\mathbf{a}, r)$ *is continuous for every* $r \in (0, r_{\mathbf{a}}) \setminus N_{\mathbf{a}}$ *and, moreover, for every* $s, t \in (0, r_{\mathbf{a}}) \setminus N_{\mathbf{a}}$ *with* $s < t$*,*

$$\mathrm{im}_T(\mathbf{u}, B(\mathbf{a}, s)) \cup \mathbf{u}(\partial B(\mathbf{a}, s)) \subset \mathrm{im}_T(\mathbf{u}, B(\mathbf{a}, t)) \cup \mathbf{u}(\partial B(\mathbf{a}, t)). \tag{1.48}$$

1.3.6 The Distributional Jacobian

If $\mathbf{u} \in W^{1,p}_{loc}(D; \mathbb{R}^n)$, with $p \geq \frac{n^2}{n+1}$, then the linear functional $(\mathrm{Det}\nabla\mathbf{u}) : C_0^\infty(D) \to \mathbb{R}$ given by

$$(\mathrm{Det}\nabla\mathbf{u})(\phi) := -\frac{1}{n}\int_D \nabla\phi \cdot (\mathrm{adj}\nabla\mathbf{u})\mathbf{u}\, d\mathbf{x} \qquad \text{for all } \phi \in C_0^\infty(D) \tag{1.49}$$

is a well defined distribution, which is called the **distributional Jacobian.** If $\mathbf{u} \in W^{1,p}_{loc}(D; \mathbb{R}^n)$, with $p \geq n$ then the identity $\mathrm{Div}(\mathrm{adj}\nabla\mathbf{u})^T = 0$ can be used to show that $\mathrm{Det}\nabla\mathbf{u}$ is the distribution induced by the function $\det \nabla\mathbf{u}$. (In general this need not be the case and in fact it will not be when cavitation occurs.)

Now suppose that $\mathbf{u} \in W^{1,p}_{loc}(D; \mathbb{R}^n)$, with $p > n - 1$. Then, for every $\mathbf{a} \in D$, the precise representative $\mathbf{u}^*$ is continuous on the sphere $\partial B(\mathbf{a}, r)$ for almost every $r \in (0, r_{\mathbf{a}})$ and hence $\mathbf{u}^*(\partial B(\mathbf{a}, r))$ is compact for such r. If, in addition, $\mathbf{u}^*$ satisfies condition (INV) then it follows that $\mathbf{u}^* \in L^\infty_{loc}(D; \mathbb{R}^n)$ and hence that the above functional is once again a well-defined distribution on D. The next result shows that in fact this distribution is a non-negative Radon measure.

Proposition 1.38. ([34, Lemma 8.1]). *Let* $\mathbf{u} \in W^{1,p}_{loc}(D; \mathbb{R}^n)$*, with* $p > n - 1$*. Suppose that* $\det \nabla\mathbf{u} > 0$ *a.e. and that* $\mathbf{u}^*$ *satisfies condition* (INV)*. Then* $\mathrm{Det}\,\nabla\mathbf{u} \geq 0$ *(in the sense that* $(\mathrm{Det}\,\nabla\mathbf{u})(\phi) \geq 0$ *whenever* $\phi \geq 0$*), and hence*

$\operatorname{Det} \nabla \mathbf{u}$ *is a Radon measure. Furthermore,*

$$\operatorname{Det} \nabla \mathbf{u} = (\det \nabla \mathbf{u})\mathcal{L}^n + m_{\mathbf{u}}, \tag{1.50}$$

where $m_{\mathbf{u}}$ *is singular with respect to Lebesgue measure and for* $\mathcal{L}^1$ *a.e.* $r \in (0, r_{\mathbf{a}})$ *one has*

$$(\operatorname{Det} \nabla \mathbf{u})(B(\mathbf{a}, r)) = \mathcal{L}^n(\operatorname{im}_T(\mathbf{u}, B(\mathbf{a}, r))). \tag{1.51}$$

It is sometimes necessary to consider the homogeneous extension of mappings satisfying affine boundary conditions. For every deformation $\mathbf{u}$ such that $\mathbf{u} - \mathbf{A}\mathbf{x} \in W_0^{1,p}(\Omega; \mathbb{R}^n)$, we denote by $\mathbf{u}^e$ its homogeneous extension to all of $\mathbb{R}^n$ as the linear deformation $\mathbf{A}\mathbf{x}$, i.e. $\mathbf{u}^e : \mathbb{R}^n \to \mathbb{R}^n$ is given by

$$\mathbf{u}^e(\mathbf{x}) = \begin{cases} \mathbf{u}^*(\mathbf{x}), & \mathbf{x} \in \Omega, \\ \mathbf{A}\mathbf{x}, & \mathbf{x} \notin \Omega. \end{cases} \tag{1.52}$$

Note that $\mathbf{u}^e \in W_{loc}^{1,p}(\mathbb{R}^n; \mathbb{R}^n)$.

The following relation will be extensively used in this thesis. Let $\mathcal{A}_{\mathbf{A},p}$ be the class of deformations which is analogous for general n to that defined in (1.22) for $n = 3$. If $\mathbf{u} \in \mathcal{A}_{\mathbf{A},p}$, then

$$\int_\Omega [\det \mathbf{A} - \det \nabla \mathbf{u}]\, d\mathbf{x} = m_{\mathbf{u}}(\overline{\Omega}). \tag{1.53}$$

To see this, let R_0 be such that $\Omega \subset B(\mathbf{0}, R_0)$, and let R_1 be such that $R_1 > R_0$ and (1.51) holds. Then, by (1.50),

$$\operatorname{Det} \nabla \mathbf{u}^e = (\det \nabla \mathbf{u}^e)\mathcal{L}^n + m_{\mathbf{u}},$$

where $m_{\mathbf{u}} \geq 0$ is a Radon measure which is singular with respect to Lebesgue measure. Since $\Omega \subset\subset B(\mathbf{0}, R_1)$, the definition of the topological image and (1.52) imply that

$$\operatorname{im}_T(\mathbf{u}^e, B(\mathbf{0}, R_1)) = \mathbf{A}B(\mathbf{0}, R_1).$$

Thus, if we evaluate $\operatorname{Det} \nabla \mathbf{u}^e$ on the ball $B(\mathbf{0}, R_1)$ and make use of (1.50) and

(1.51), we find that

$$
\begin{aligned}
&(\det \mathbf{A})\mathcal{L}^n(B(\mathbf{0}, R_1)) \\
&\quad = (\mathrm{Det}\nabla \mathbf{u}^e)(B(\mathbf{0}, R_1)) \\
&\quad = m_{\mathbf{u}}(B(\mathbf{0}, R_1)) + \int_{B(\mathbf{0},R_1)} \det \nabla \mathbf{u}^e(\mathbf{x})\, d\mathbf{x} \\
&\quad = m_{\mathbf{u}}(\overline{\Omega}) + (\det \mathbf{A})[\mathcal{L}^n(B(\mathbf{0}, R_1)) - \mathcal{L}^n(\Omega)] + \int_{\Omega} \det \nabla \mathbf{u}(\mathbf{x})\, d\mathbf{x},
\end{aligned}
$$

since $\nabla \mathbf{u}^e = \mathbf{A}$ on $B(\mathbf{0}, R_1) \setminus \Omega$ and the support of $m_{\mathbf{u}}$ is contained in $\overline{\Omega}$. If we rearrange terms, we find that (1.53) holds.

1.3.7 Radial Cavitation

In this section we discuss certain properties of radial cavitation solutions found in [8]. We consider deformations of a homogeneous ball of elastic material which in its reference configuration occupies the region $\Omega = B$, the unit ball in $\mathbb{R}^3$. We restrict attention to **radial** deformations $\mathbf{u}$, i.e. deformations of the form

$$
\mathbf{u}(\mathbf{x}) = r(|\mathbf{x}|)\frac{\mathbf{x}}{|\mathbf{x}|}. \tag{1.54}
$$

The following result relates the properties of $\mathbf{u}$ and r.

Proposition 1.39. ([8, Lemma 4.1]). *Let $1 \leq p < \infty$ and let $\mathbf{u}$ be given by (1.54). Then $\mathbf{u} \in W^{1,p}(\Omega; \mathbb{R}^3)$ if and only r is absolutely continuous on $(0, 1)$ and*

$$
\int_0^1 R^2 \left[|r'(R)|^p + \left| \frac{r(R)}{R} \right|^p \right] dR < +\infty.
$$

In this case, the weak derivatives of $\mathbf{u}$ are given by

$$
\nabla \mathbf{u}(\mathbf{x}) = \frac{r(R)}{R}\mathbf{I} + \left(r'(R) - \frac{r(R)}{R} \right) \frac{\mathbf{x} \otimes \mathbf{x}}{R^2} \quad \text{a.e. } x \in B. \tag{1.55}
$$

The condition

$$
\det \nabla \mathbf{u}(\mathbf{x}) > 0 \quad \text{for almost every } \mathbf{x} \in B \tag{1.56}
$$

is easily seen to be equivalent to

$$r'(R) > 0 \quad \text{for almost every } R \in (0,1). \tag{1.57}$$

In the displacement boundary value problem the values of $\mathbf{u}$ are prescribed on the boundary of B,

$$\mathbf{u}(\mathbf{x}) = \lambda \mathbf{x} \quad \text{for all } \mathbf{x} \in \partial B, \tag{1.58}$$

which is equivalent to the condition $r(1) = \lambda$.

Example. Let $\mathbf{u} \in W^{1,p}(B;\mathbb{R}^3)$, $p \in (2,3)$, be a radial map such that (1.56) holds. Then $\mathbf{u}$ satisfies condition (INV). Moreover, the singular measure $m_{\mathbf{u}}$ in (1.50) satisfies

$$m_{\mathbf{u}} = \frac{4\pi}{3} r^3(0)\delta_{\mathbf{0}},$$

where $\delta_{\mathbf{0}}$ is the Dirac measure supported at $\mathbf{0}$.

If W is frame-indifferent and isotropic (i.e., if (1.3) and (1.4) hold), then it is well-known that there exists a symmetric function Φ such that

$$W(\mathbf{F}) = \Phi(v_1, v_2, v_3) \qquad \text{for all } \mathbf{F} \in M^{3\times 3}_{+}, \tag{1.59}$$

where v_1, v_2, v_3, known as the **principal stretches** of $\mathbf{F}$, are the eigenvalues of $(\mathbf{F}^T\mathbf{F})^{1/2}$.

In the case of a radial deformation $\mathbf{u}$, the principal stretches of the matrix $\mathbf{F} = \nabla \mathbf{u}(\mathbf{x})$ are given by

$$v_1 = r'(R), \;\; v_2 = v_3 = \frac{r(R)}{R} \qquad \text{where } R = |\mathbf{x}|.$$

By (1.2) and (1.59), the corresponding energy takes the form

$$E(\mathbf{u}) = 4\pi I(r) := 4\pi \int_0^1 R^2 \Phi\left(r'(R), \frac{r(R)}{R}, \frac{r(R)}{R}\right) dR. \tag{1.60}$$

It is shown in [8, Theorem 4.2] that the study of weak solutions to (1.9) of the form (1.54) is equivalent to the study of weak solutions r on $(0,1)$ of the **radial**

equilibrium equations

$$\frac{d}{dR}\left[R^2\Phi_1\left(r'(R),\frac{r(R)}{R},\frac{r(R)}{R}\right)\right]=2R\Phi_2\left(r'(R),\frac{r(R)}{R},\frac{r(R)}{R}\right). \tag{1.61}$$

Under very general assumptions on Φ, weak solutions of (1.61) are shown in [8, Proposition 6.1] to be classical solutions, i.e. $r \in C^2((0,1])$ and (1.61) holds everywhere on $(0,1]$. Note that (1.61) can also be written in the form

$$\begin{aligned}R\frac{d}{dR}&\left[\Phi_1\left(r'(R),\frac{r(R)}{R},\frac{r(R)}{R}\right)\right]\\&=2\left[\Phi_2\left(r'(R),\frac{r(R)}{R},\frac{r(R)}{R}\right)-\Phi_1\left(r'(R),\frac{r(R)}{R},\frac{r(R)}{R}\right)\right].\end{aligned} \tag{1.62}$$

To demonstrate the existence of non-trivial solutions of (1.61) corresponding to cavitation, Ball [8, Theorem 7.1 and Theorem 7.2] used a variational technique, showing that the functional I attains its infimum on a set of admissible functions A_λ, where

$$A_\lambda=\{r\in W^{1,1}(0,1): r(1)=\lambda, r'(R)>0 \text{ a.e. } R\in(0,1), r(0)\geq 0\}. \tag{1.63}$$

He showed that any such minimiser is a classical solution of (1.61) and, in addition,

$$\text{if } r(0):=\lim_{R\to 0} r(R)>0 \quad\text{then}\quad \lim_{R\to 0} T(r(R))=0 \tag{1.64}$$

where

$$T(r(R)):=\left(\frac{R}{r(R)}\right)^2\Phi_1\left(r'(R),\frac{r(R)}{R},\frac{r(R)}{R}\right)$$

is the radial component of the Cauchy stress. It follows from (1.54) that if $r(0)>0$ then the deformed ball contains a cavity and (1.64) is the natural boundary condition that the cavity is stress free. It was also shown in [8] that, for sufficiently large values of the boundary displacement λ, the minimiser r of I on A_λ necessarily satisfies $r(0)>0$.

It is sometimes of interest to consider solutions of the radial Euler-Lagrange equations on shells. The following result, which will be used later, shows that solutions of the radial problem generate solutions of the full three-dimensional problem.

Proposition 1.40. *Let $\varepsilon \in (0,1)$, and let $r \in C^2([\varepsilon,1])$, with $r(\varepsilon) > 0$, satisfy (1.61) on $[\varepsilon, 1]$, and be such that $T(r(\varepsilon)) = 0$. Then, for all $\varphi \in W^{1,1}(B \setminus B_\varepsilon; \mathbb{R}^3)$ with $\varphi = 0$ on ∂B,*

$$\int_{B\setminus B_\varepsilon} \frac{\partial W}{\partial \mathbf{F}}(\nabla \mathbf{u}) : \nabla \varphi \, d\mathbf{x} = 0. \tag{1.65}$$

Proof of Proposition 1.40. As shown by Ball [8], every C^2 radial mapping $\mathbf{u}$ of the form (1.54) satisfies

$$\begin{aligned}\frac{\partial W}{\partial \mathbf{F}^i_\alpha}(\nabla \mathbf{u}(\mathbf{x})) &= \Phi_1 \left(r'(R), \frac{r(R)}{R}, \frac{r(R)}{R} \right) \cdot \frac{x_i x_\alpha}{R^2} \\ &\quad + \Phi_2 \left(r'(R), \frac{r(R)}{R}, \frac{r(R)}{R} \right) \left(\delta^i_\alpha - \frac{x_i x_\alpha}{R^2} \right),\end{aligned} \tag{1.66}$$

for all $\mathbf{x} \in B_1 \setminus B_\varepsilon$ and $i, \alpha \in \{1,2,3\}$. It follows that, for every fixed $i \in \{1,2,3\}$,

$$\frac{\partial}{\partial x_\alpha} \left(\frac{\partial W}{\partial \mathbf{F}^i_\alpha}(\nabla \mathbf{u}(\mathbf{x})) \right) = \frac{x_i}{R} \left[\frac{d}{dR}(R^2 \Phi_1) - 2R\Phi_2 \right]. \tag{1.67}$$

Also, a simple calculation using (1.66) shows that

$$\frac{\partial W}{\partial \mathbf{F}^i_\alpha}(\nabla \mathbf{u}(\mathbf{x})) n_\alpha = \frac{x_i}{R} \Phi_1 \left(r'(R), \frac{r(R)}{R}, \frac{r(R)}{R} \right), \tag{1.68}$$

where $\mathbf{n} = (n_1, n_2, n_3)$ is the normal to ∂B_R.

Suppose now that r is as in the statement of the proposition. An integration by parts yields

$$\begin{aligned}\int_{B\setminus B_\varepsilon} \frac{\partial W}{\partial \mathbf{F}}(\nabla \mathbf{u}) : \nabla \varphi \, d\mathbf{x} &= \int_{B\setminus B_\varepsilon} \frac{\partial W}{\partial \mathbf{F}^i_\alpha}(\nabla \mathbf{u}) \frac{\partial \varphi_i}{\partial x_\alpha} \, d\mathbf{x} \\ &= - \int_{B\setminus B_\varepsilon} \frac{\partial}{\partial x_\alpha} \left(\frac{\partial W}{\partial \mathbf{F}^i_\alpha}(\nabla \mathbf{u}) \right) \varphi_i \, d\mathbf{x} \\ &\quad + \int_{\partial B} \frac{\partial W}{\partial \mathbf{F}^i_\alpha}(\nabla \mathbf{u}) \varphi_i n_\alpha \, d\mathcal{H}^2 + \int_{\partial B_\varepsilon} \frac{\partial W}{\partial \mathbf{F}^i_\alpha}(\nabla \mathbf{u}) \varphi_i n_\alpha \, d\mathcal{H}^2.\end{aligned}$$

Since $\varphi = 0$ on ∂B, r satisfies (1.61) on the interval $[\varepsilon, 1]$ and the natural boundary condition that the cavity surface is stress free, the required conclusion follows using (1.67) and (1.68). □

We now consider properties of solutions of (1.61). It is always assumed that

$$\Phi_{11}(q,t,t) > 0 \quad \text{for all } q,t \in (0,\infty). \tag{1.69}$$

Proposition 1.41. ([42, Proposition 1.1 and Corollary 1.2]). *Let r be a solution of* (1.61) *on an interval J of $(0,\infty)$, where Φ satisfies* (1.69). *If $R \mapsto r(R)/R$ is not constant on J, then $r'(R) \neq r(R)/R$ for all $R \in J$. Moreover, $R \mapsto r(R)/R$ is strictly monotone on J. In particular, if $J = (0,a]$ and $r(0) = \lim_{R \searrow 0} r(R) > 0$, then $r'(R) < r(R)/R$ for all $R \in (0,a]$.*

Let us now consider the following constitutive assumption

$$\frac{\Phi_2 - \Phi_1}{q-t} - \Phi_{12} < 0 \quad \text{for all } q,t \in (0,\infty), \tag{1.70}$$

where the partial derivatives of Φ are evaluated at (q,t,t).

Proposition 1.42. ([42, Proposition 1.5]). *If Φ satisfies* (1.69) *and* (1.70), *and r is a solution of* (1.61) *on an interval J of $(0,\infty)$, then*

$$r''(R)\left(r'(R) - \frac{r(R)}{R}\right) \leq 0 \quad \textit{for all } R \in J.$$

In his study of solutions of (1.61), Stuart [49] noted that, if one makes the change of variables

$$t = \frac{r(R)}{R}, \quad q(t) = r'(R), \tag{1.71}$$

then q solves the ordinary differential equation

$$q'(t) = \frac{2}{\Phi_{11}}\left(\frac{\Phi_2 - \Phi_1}{q-t} - \Phi_{12}\right), \tag{1.72}$$

in which the derivatives of Φ are evaluated at $(q(t),t,t)$. This transformation will play an important role in Chapter 3 of this thesis.

For $q \neq t$, $q,t \in (0,\infty)$, let

$$R(q,t) = \frac{q\Phi_1(q,t,t) - t\Phi_2(q,t,t)}{q-t}. \tag{1.73}$$

Then R has a C^1 extension to $(0,\infty)^2$. Consider the constitutive assumption

$$\frac{\partial R}{\partial q}(q,t) \geq 0 \quad \text{for all } 0 < q \leq t. \tag{1.74}$$

Proposition 1.43. ([50, Theorem 3.1 and Corollary 3.2]). *Suppose that Φ satisfies* (1.69), (1.70) *and* (1.74). *Let r be a solution of* (1.61) *on an interval J of* $(0,\infty)$ *such that $r'(R) < r(R)/R$ for all $R \in J$. Then the function $d : J \to \mathbb{R}$ given by*

$$d(R) = r'(R)\left(\frac{r(R)}{R}\right)^2 \quad \textit{for all } R \in J$$

is an increasing function on J.

The following conservation law from [8] and [42], which is satisfied by solutions of the Euler-Lagrange equations will be used in this thesis.

Proposition 1.44. *If $r \in C^2((0,1])$ is a solution of* (1.61) *with $r'(R) > 0$ for $R \in (0,1]$ then*

$$\frac{d}{dR}\left\{R^3\left[\Phi\left(r'(R),\frac{r(R)}{R},\frac{r(R)}{R}\right)+\left(\frac{r(R)}{R}-r'(R)\right)\Phi_1\left(r'(R),\frac{r(R)}{R},\frac{r(R)}{R}\right)\right]\right\}$$
$$= 3R^2\Phi\left(r'(R),\frac{r(R)}{R},\frac{r(R)}{R}\right), \tag{1.75}$$

for $R \in (0,1]$.

Remark 1.45. It was noted by Ball [8] that (1.75) is the specialisation to radial solutions of the 3-dimensional conservation law (originally due to Green [20])

$$\sum_\alpha \frac{\partial}{\partial x^\alpha}\left[x^\alpha W(\nabla\mathbf{u}) + (u^i - \frac{\partial u^i}{\partial x_k}x^k)\frac{\partial W}{\partial F^i_\alpha}(\nabla\mathbf{u})\right] = 3W(\nabla\mathbf{u}). \tag{1.76}$$

Equation (1.76) was also used by Knops and Stuart [28] to prove the uniqueness of smooth equilibrium solutions to the displacement boundary value problem of elasticity for star-shaped domains (see Chapter 6 of this thesis for further details).

1.4 Elementary facts about $W^{1,p}$-quasiconvexity

In this thesis we will be mainly interested in the $W^{1,p}$-quasiconvexity of stored energy functions over various classes of deformations. In doing this we will use the following notation.

Definition 1.46. *Let $p \geq 1$. Given a stored energy function W, a matrix $\mathbf{A} \in M_+^{n\times n}$ and a class of mappings $\mathcal{A}(\Omega)$ contained in $W^{1,p}(\Omega;\mathbb{R}^n)$, we say that W is $W^{1,p}$-quasiconvex at $\mathbf{A}$ over $\mathcal{A}(\Omega)$ if*

$$E(\mathbf{u}_{\mathbf{A}}^{\mathrm{hom}}) \leq E(\mathbf{u}) \quad \textit{for all } \mathbf{u} \in \mathcal{A}(\Omega) \textit{ with } \mathbf{u}(\mathbf{x}) = \mathbf{A}\mathbf{x} \textit{ on } \partial\Omega, \tag{1.77}$$

where, for all $\mathbf{u} \in \mathcal{A}(\Omega)$,

$$E(\mathbf{u}) = \int_\Omega W(\nabla \mathbf{u}(\mathbf{x}))\, d\mathbf{x}.$$

Some classes of deformations which will be of interest, for $p \in (n-1, n)$, are

$$\begin{aligned} \mathcal{A}_{\mathbf{A},p}(\Omega) := \{\mathbf{u} \in W^{1,p}(\Omega;\mathbb{R}^n) \ : \mathbf{u} - \mathbf{A}\mathbf{x} \in W_0^{1,p}(\Omega;\mathbb{R}^n),\ \det \nabla \mathbf{u} > 0 \text{ a.e.},\\ \mathbf{u}^e \text{ satisfies (INV)}\}, \end{aligned} \tag{1.78}$$

$$\begin{aligned} \mathcal{A}_{\mathbf{A},p}(\Omega;\mathbf{a}_1,\dots,\mathbf{a}_N) := \Big\{\mathbf{u} \in \mathcal{A}_{\mathbf{A},p}(\Omega) :\ \mathrm{Det}\ \nabla\mathbf{u}^e = (\det \nabla \mathbf{u}^e)\mathcal{L}^n + \sum_{i=1}^N \alpha_i \delta_{\mathbf{a}_i},\\ \alpha_i \geq 0 \text{ for all } i = 1,\dots,N\Big\}, \end{aligned} \tag{1.79}$$

and

$$\mathcal{A}^*_{\mathbf{A},p}(\Omega) := \cup_{\mathbf{a}\in\Omega}\mathcal{A}_{\mathbf{A},p}(\Omega;\mathbf{a}). \tag{1.80}$$

The deformations in the class $\mathcal{A}^*_{\mathbf{A},p}(\Omega)$ can be interpreted as opening at most a single hole anywhere in the material.

In the study of certain stored energy functions on $M_+^{n\times n}$, such as W_h given by (1.29) with $p = 2$ for $n = 3$, it is most natural to work with mappings in a subclass of $W^{1,n-1}(\Omega;\mathbb{R}^n)$. For such mappings, the condition (INV) cannot be defined in the same way as it was for mappings in $W^{1,p}(\Omega;\mathbb{R}^n)$, $p > n-1$, due to the possible lack of of continuity of the restrictions of these mappings to spheres.

However, a definition of condition (INV) is possible, see [15], but at the expense of significant technical complications. In this thesis we take a different approach, and for such functionals we investigate their quasiconvexity over the classes

$$\widetilde{\mathcal{A}}_{\mathbf{A},n-1}(\Omega) := \cup_{n-1<p<n}\mathcal{A}_{\mathbf{A},p}(\Omega) \tag{1.81}$$

$$\widetilde{\mathcal{A}}_{\mathbf{A},n-1}(\Omega;\mathbf{a}_1,...,\mathbf{a}_N) := \cup_{n-1<p<n}\mathcal{A}_{\mathbf{A},p}(\Omega;\mathbf{a}_1,...\mathbf{a}_N) \tag{1.82}$$

$$\widetilde{\mathcal{A}}^*_{\mathbf{A},n-1}(\Omega) := \cup_{n-1<p<n}\mathcal{A}^*_{\mathbf{A},p}(\Omega). \tag{1.83}$$

Remark 1.47. We wish to emphasize that for the classes of deformations introduced in (1.78)-(1.80) the range of values of p is precisely the interval $(n-1,n)$. For the sake of simplicity and by a slight abuse of notation, we choose for the rest of the thesis to drop the 'tilde' in the notation for the classes defined in (1.81)-(1.83). Thus, for example, when a result is stated for the class $\mathcal{A}_{\mathbf{A},p}(\Omega)$, $p \in [n-1,n)$, it is meant that when $p \in (n-1,n)$ the result holds in the class defined in (1.78) while if $p = n-1$ the result holds in the class defined in (1.81).

The next result shows that the notion of $W^{1,p}$-quasiconvexity over various classes is independent of the domain.

Proposition 1.48. *Let $\mathbf{A} \in M^{n\times n}_+$ and let Ω_1, Ω_2 be bounded open sets of $\mathbb{R}^n$. Then W is $W^{1,p}$-quasiconvex at $\mathbf{A}$ over $\mathcal{A}_{\mathbf{A},p}(\Omega_1)$ if and only if W is $W^{1,p}$-quasiconvex at $\mathbf{A}$ over $\mathcal{A}_{\mathbf{A},p}(\Omega_2)$. Similarly, W is $W^{1,p}$-quasiconvex at $\mathbf{A}$ over $\mathcal{A}^*_{\mathbf{A},p}(\Omega_1)$ if and only if W is $W^{1,p}$-quasiconvex at $\mathbf{A}$ over $\mathcal{A}^*_{\mathbf{A},p}(\Omega_2)$.*

Proof of Proposition 1.48. Suppose that W is not $W^{1,p}$-quasiconvex at $\mathbf{A}$ over $\mathcal{A}_{\mathbf{A},p}(\Omega_1)$. Then there exists $\mathbf{u}_1 \in \mathcal{A}_{\mathbf{A},p}(\Omega_1)$ such that $E(\mathbf{u}_1) < E(\mathbf{u}^{\mathrm{hom}}_{\mathbf{A}})$. Let $\mathbf{a}_1 \in \Omega_1$ and $\mathbf{a}_2 \in \Omega_2$. Let $\mathbf{u}_2 : \Omega_2 \to \mathbb{R}^n$ be given by

$$\mathbf{u}_2(\mathbf{x}) = \begin{cases} \mathbf{A}\mathbf{a}_2 + \varepsilon\mathbf{u}_1\left(\mathbf{a}_1 + \dfrac{\mathbf{x}-\mathbf{a}_2}{\varepsilon}\right), & \text{if } \mathbf{x} \in \mathbf{a}_2 + \varepsilon(\Omega_1 - \mathbf{a}_1), \\ \mathbf{A}\mathbf{x}, & \text{otherwise}. \end{cases} \tag{1.84}$$

Then it is easily verified that, for small ε, $\mathbf{u}_2 \in \mathcal{A}_{\mathbf{A},p}(\Omega_2)$ and

$$
\begin{aligned}
E(\mathbf{u}_2) - E(\mathbf{u}_{\mathbf{A}}^{\text{hom}}) &= \int_{\mathbf{a}_2+\varepsilon(\Omega_1-\mathbf{a}_1)} \left[W\left(\nabla \mathbf{u}_1\left(\mathbf{a}_1 + \frac{\mathbf{x}-\mathbf{a}_2}{\varepsilon}\right)\right) - W(\mathbf{A})\right] d\mathbf{x} \\
&= \varepsilon^n \int_{\Omega} [W(\nabla \mathbf{u}_1(\mathbf{y})) - W(\mathbf{A})]\, d\mathbf{y} \\
&= \varepsilon^n [E(\mathbf{u}_1) - E(\mathbf{u}_{\mathbf{A}}^{\text{hom}})] < 0. \qquad (1.85)
\end{aligned}
$$

Hence W is not $W^{1,p}$-quasiconvex at $\mathbf{A}$ over $\mathcal{A}_{\mathbf{A},p}(\Omega_2)$. Since the roles of Ω_1 and Ω_2 in the above argument can be interchanged, the first part of the conclusion follows. The second part of the required conclusion follows by the same argument.

□

The same argument can be used to prove the following result from [44].

Proposition 1.49. *Let $\mathbf{A} \in M_+^{n\times n}$ and let Ω be a bounded open subset of $\mathbb{R}^n$. Then for any $\mathbf{a}_1$, $\mathbf{a}_2 \in \Omega$, W is $W^{1,p}$-quasiconvex over $\mathcal{A}_{\mathbf{A},p}(\Omega;\mathbf{a}_1)$ if and only if W is $W^{1,p}$-quasiconvex over $\mathcal{A}_{\mathbf{A},p}(\Omega;\mathbf{a}_2)$. Hence, for any $\mathbf{a} \in \Omega$, W is $W^{1,p}$-quasiconvex at $\mathbf{A}$ over $\mathcal{A}_{\mathbf{A},p}(\Omega;\mathbf{a})$ if and only if W is $W^{1,p}$-quasiconvex at $\mathbf{A}$ over $\mathcal{A}^*_{\mathbf{A},p}(\Omega)$.*

The next result concerns the $W^{1,p}$-quasiconvexity of stored energy functions which are frame-indifferent and isotropic, as defined in (1.3) and (1.4). It shows that for the study of $W^{1,p}$-quasiconvexity it is necessary and sufficient to consider only diagonal matrices.

Proposition 1.50. *Let $\mathbf{A} \in M_+^{n\times n}$ with principal stretches $\lambda_1, \lambda_2, ..., \lambda_n$ and let $\mathbf{D} = diag(\lambda_1, \lambda_2, ..., \lambda_n)$. Let Ω be a bounded open set in $\mathbb{R}^n$, and let W satisfy* (1.3) *and* (1.4). *Then W is $W^{1,p}$-quasiconvex at $\mathbf{A}$ over $\mathcal{A}_{\mathbf{A},p}(\Omega)$ if and only if W is $W^{1,p}$-quasiconvex at $\mathbf{D}$ over $\mathcal{A}_{\mathbf{D},p}(\Omega)$. Similarly, W is $W^{1,p}$-quasiconvex at $\mathbf{A}$ over $\mathcal{A}^*_{\mathbf{A},p}(\Omega)$ if and only if W is $W^{1,p}$-quasiconvex at $\mathbf{D}$ over $\mathcal{A}^*_{\mathbf{D},p}(\Omega)$.*

Proof of Proposition 1.50. We prove the equivalence between the $W^{1,p}$-quasiconvexity at $\mathbf{A}$ over $\mathcal{A}_{\mathbf{A},p}(\Omega)$ and the $W^{1,p}$-quasiconvexity at $\mathbf{D}$ over $\mathcal{A}_{\mathbf{D},p}(\Omega)$. Very similar arguments can be used to prove the equivalence between the $W^{1,p}$-quasiconvexity at $\mathbf{A}$ over $\mathcal{A}^*_{\mathbf{A},p}(\Omega)$ the $W^{1,p}$-quasiconvexity at $\mathbf{D}$ over $\mathcal{A}^*_{\mathbf{D},p}(\Omega)$.

By Proposition 1.48, the $W^{1,p}$-quasiconvexity property does not depend on the domain, so there is no loss of generality in assuming that $\Omega := B(\mathbf{0}, 1)$, the unit ball in $\mathbb{R}^n$.

Suppose that W is not $W^{1,p}$-quasiconvex at $\mathbf{A}$ over $\mathcal{A}_{\mathbf{A},p}(\Omega)$. Then there exists a mapping $\mathbf{u} \in \mathcal{A}_{\mathbf{A},p}(\Omega)$ such that $E(\mathbf{u}) < E(\mathbf{u}_{\mathbf{A}}^{\text{hom}})$. We shall construct a mapping $\tilde{\mathbf{u}} \in \mathcal{A}_{\mathbf{D},p}(\Omega)$ such that $E(\tilde{\mathbf{u}}) < E(\mathbf{u}_{\mathbf{D}}^{\text{hom}})$. By standard results on the factorisation of matrices, see [14, Theorem 3.2-3, p.98], there exist matrices $\mathbf{P}, \mathbf{Q} \in SO(n)$ such that $\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{Q}^{\mathrm{T}}$ and $\mathbf{D} = \mathbf{P}^{\mathrm{T}}\mathbf{A}\mathbf{Q}$.

Let us consider the mapping $\tilde{\mathbf{u}} \in W^{1,p}(\Omega; \mathbb{R}^n)$ given by $\tilde{\mathbf{u}}(\mathbf{x}) := \mathbf{P}^{\mathrm{T}}\mathbf{u}(\mathbf{Q}\mathbf{x})$, for all $\mathbf{x} \in \Omega$. Then $\mathbf{x} \in \partial\Omega$ if and only if $\mathbf{Q}\mathbf{x} \in \partial\Omega$. ($\Omega$ is the unit ball, and hence is rotationally symmetric.) Therefore,

$$\tilde{\mathbf{u}}(\mathbf{x}) = \mathbf{P}^{\mathrm{T}}\mathbf{u}(\mathbf{Q}\mathbf{x}) = \mathbf{P}^{\mathrm{T}}\mathbf{A}\mathbf{Q}\mathbf{x} = \mathbf{D}\mathbf{x} \qquad \text{for all } \mathbf{x} \in \partial\Omega.$$

Taking into account the relations (1.3) and (1.4), we obtain

$$\begin{aligned} E(\tilde{\mathbf{u}}) &= \int_\Omega W(\nabla\tilde{\mathbf{u}}(\mathbf{x}))\, d\mathbf{x} \\ &= \int_\Omega W(\mathbf{P}^{\mathrm{T}}\nabla\mathbf{u}(\mathbf{Q}\mathbf{x})\mathbf{Q})\, d\mathbf{x} = \int_\Omega W(\nabla\mathbf{u}(\mathbf{Q}\mathbf{x}))\, d\mathbf{x} \\ &= \int_\Omega W(\nabla\mathbf{u}(\mathbf{y}))\, d\mathbf{y} = E(\mathbf{u}). \end{aligned}$$

It is straightforward from (1.3) and (1.4) that $E(\mathbf{u}_{\mathbf{D}}^{\text{hom}}) = E(\mathbf{u}_{\mathbf{A}}^{\text{hom}})$, since $W(\mathbf{D}) = W(\mathbf{P}^{\mathrm{T}}\mathbf{A}\mathbf{Q}) = W(\mathbf{A})$. Therefore $E(\tilde{\mathbf{u}}) < E(\mathbf{u}_{\mathbf{D}}^{\text{hom}})$ and hence, since $\tilde{\mathbf{u}} \in \mathcal{A}_{\mathbf{D},p}(\Omega)$, it follows that W is not $W^{1,p}$-quasiconvex at $\mathbf{D}$ over $\mathcal{A}_{\mathbf{D},p}(\Omega)$.

By similar arguments, one can prove that, if W is not $W^{1,p}$-quasiconvex at $\mathbf{D}$ over $\mathcal{A}_{\mathbf{D},p}(\Omega)$, then W is not $W^{1,p}$-quasiconvex at $\mathbf{A}$ over $\mathcal{A}_{\mathbf{A},p}(\Omega)$. This completes the proof of Proposition 1.50. $\square$

Remark 1.51. For the sake of simplicity, for the rest of the thesis we suppress the dependence on Ω in the notation (the specific domain Ω to which these classes refer will bé clear from the context) in (1.78)-(1.80) and (1.81)-(1.83). We denote those classes by $\mathcal{A}_{\mathbf{A},p}$, $\mathcal{A}_{\mathbf{A},p}(\mathbf{a}_1, ..., \mathbf{a}_N)$ and $\mathcal{A}_{\mathbf{A},p}^{*}$ respectively.

Chapter 2

Sufficient Conditions for $W^{1,p}$-quasiconvexity

In this chapter we study sufficient conditions for $W^{1,p}$-quasiconvexity for the stored energy function given by

$$W_h(\mathbf{F}) = |\mathbf{F}|^p + h(\det \mathbf{F}) \qquad \text{for all } \mathbf{F} \in M_+^{3\times 3}, \tag{2.1}$$

where $p \in [2,3)$ and h satisfies (1.30).

In [35], Müller, Spector and Sivaloganathan proved that there exists a constant $k > 0$ such that if

$$h'(\det \mathbf{A})|\mathbf{A}|^{3-p} \leq k, \tag{2.2}$$

then W_h is $W^{1,p}$-quasiconvex at $\mathbf{A}$ over $\mathcal{A}_{\mathbf{A},p}$. This was an improvement of a result of Spector [48], who had proved that if $\mathbf{A}$ is such that

$$h'(\det \mathbf{A}) \leq 0,$$

then W_h is $W^{1,p}$-quasiconvex at $\mathbf{A}$ over a class of deformations slightly more general than $\mathcal{A}_{\mathbf{A},p}$, see Section 1.2 for details.

The most important relation for proving the result in [35] is an isoperimetric estimate that bounds the integral of the difference of the Jacobians of $\mathbf{Ax}$ and $\mathbf{u}$ in terms of the L^p norm of the difference of their gradients. That is, for every $n \geq 2$ and $p \in (n-1, n)$ there is a constant $\mu = \mu(n,p)$ such that for every n by

n matrix $\mathbf{A}$ with positive determinant and for every bounded open set $\Omega \subset \mathbb{R}^n$

$$\int_\Omega [\det \mathbf{A} - \det \nabla \mathbf{u}(\mathbf{x})]\, d\mathbf{x} \le \mu |\mathbf{A}|^{n-p} \int_\Omega |\mathbf{A} - \nabla \mathbf{u}(\mathbf{x})|^p \, d\mathbf{x}, \tag{2.3}$$

for all deformations $\mathbf{u} \in \mathcal{A}_{\mathbf{A},p}$. Although it is stated in [35] that it would be of interest to give numerical bounds on the value of the constant μ in (2.3), no such estimate is given there.

Here we obtain an explicit value of the constant μ in (2.3), which significantly improves the value which could be obtained by the arguments in [35]. This in turn leads to a better value of the constant k in (2.2).

We also prove here that there exists $\bar{k} > 0$ such that, if

$$h'(\det \mathbf{A})(\det \mathbf{A})^{(3-p)/3} \le \bar{k}, \tag{2.4}$$

then W_h is $W^{1,p}$-quasiconvex at $\mathbf{A}$ over the class $\mathcal{A}^*_{\mathbf{A},p}$ of deformations opening a single hole anywhere in the material.

2.1 Sufficient condition for $W^{1,p}$-quasiconvexity of W_h, depending on the determinant and the norm of the matrix A

Let us briefly recall the basic idea of the result of Spector [48], but restricting attention to mappings in $\mathcal{A}_{\mathbf{A},p}$. Consider the case of stored energy function of the form

$$W(\mathbf{F}) = \hat{g}(\mathbf{F}, \operatorname{adj} \mathbf{F}) + h(\det \mathbf{F}), \tag{2.5}$$

where $\hat{g} : M^{3\times 3} \times M^{3\times 3} \to [0, \infty)$ is convex, and h satisfies (1.30). If we denote

$$W_0(\mathbf{F}) = \hat{g}(\mathbf{F}, \operatorname{adj} \mathbf{F}),$$

it is shown in [48] and [35] that W_0 is $W^{1,p}$-quasiconvex at every matrix $\mathbf{A}$. If W is as in (2.5), then the convexity of h and (1.53) imply that the energy E given by (1.2) satisfies, for every $\mathbf{u} \in \mathcal{A}_{\mathbf{A},p}$,

$$\begin{aligned} E(\mathbf{u}) - E(\mathbf{u}_{\mathbf{A}}^{\mathrm{hom}}) &= \int_{\Omega} W_0(\nabla\mathbf{u}) - W_0(\mathbf{A}) + h(\det\nabla\mathbf{u}) - h(\det\mathbf{A})\, d\mathbf{x} \\ &\geq h'(\det\mathbf{A}) \int_{\Omega} \det\nabla\mathbf{u} - \det\mathbf{A}\, d\mathbf{x} \\ &= -h'(\det\mathbf{A}) m_{\mathbf{u}}(\overline{\Omega}). \end{aligned} \tag{2.6}$$

Spector's result [48] is now immediate. Note also that, for any matrix $\mathbf{A}$, every $\mathbf{u}$ such that the associated singular measure $m_{\mathbf{u}}$ is identically 0 cannot have less energy than $\mathbf{u}_{\mathbf{A}}^{\mathrm{hom}}$. Hence, if W is not $W^{1,p}$-quasiconvex at $\mathbf{A}$, then, for every $\mathbf{u}$ with less energy than $\mathbf{u}_{\mathbf{A}}^{\mathrm{hom}}$, its singular measure $m_{\mathbf{u}}$ must be non-trivial.

We now determine a value of the constant μ in (2.3).

Theorem 2.1. *Let $n \geq 2$ and $p \in [n-1, n)$. Then, for any bounded open set $\Omega \subset \mathbb{R}^n$, any $\mathbf{A} \in M_{+}^{n\times n}$, and any $\mathbf{u} \in \mathcal{A}_{\mathbf{A},p}$, the following inequality holds*

$$\int_{\Omega} [\det\mathbf{A} - \det\nabla\mathbf{u}(\mathbf{x})]\, d\mathbf{x} \leq \mu|\mathbf{A}|^{n-p} \int_{\Omega} \Big||\mathbf{A}| - |\nabla\mathbf{u}(\mathbf{x})|\Big|^p d\mathbf{x}, \tag{2.7a}$$

and hence

$$\int_{\Omega} [\det\mathbf{A} - \det\nabla\mathbf{u}(\mathbf{x})]\, d\mathbf{x} \leq \mu|\mathbf{A}|^{n-p} \int_{\Omega} |\mathbf{A} - \nabla\mathbf{u}(\mathbf{x})|^p\, d\mathbf{x}, \tag{2.7b}$$

with

$$\mu = \mu(n,p) := \varpi_n (n-1)^{-n/2} \frac{n^n}{p^p (n-p)^{n-p}},$$

where ϖ_n is the constant given by Besicovitch Covering Theorem 1.21. If $\mathbf{u}$ satisfies (1.25) *with $N \leq \varpi_n$, then the above inequalities hold with*

$$\mu := N(n-1)^{-n/2} \frac{n^n}{p^p (n-p)^{n-p}}.$$

Remark 2.2. The constant obtained in Theorem 2.1 is better than the constant that would have been obtained following the arguments in [35], where the factor

$$\frac{n^n}{p^p (n-p)^{n-p}}$$

would be replaced by

$$\left(\frac{4^n}{n}\right)^{n/(n-1)}.$$

To apply Theorem 2.1 to the study of $W^{1,p}$-quasiconvexity of W_h given by (2.1), we argue as in [35]. It is proved there that there exists a positive constant K such that, for all $\mathbf{u} \in W^{1,p}(\Omega;\mathbb{R}^3)$ with $\mathbf{u}(\mathbf{x}) = \mathbf{A}\mathbf{x}$ on $\partial\Omega$,

$$\int_\Omega |\nabla\mathbf{u}|^p - |\mathbf{A}|^p\,d\mathbf{x} \geq K\int_\Omega |\nabla\mathbf{u} - \mathbf{A}|^p\,d\mathbf{x}. \tag{2.8}$$

The following result is immediate from Theorem 2.1, using (2.8) and the convexity of h.

Theorem 2.3. *Let $k := K/\mu$, where K is as in* (2.8) *and μ is as in Theorem 2.1. If $\mathbf{A} \in M_+^{3\times 3}$ satisfies*

$$h'(\det\mathbf{A})|\mathbf{A}|^{3-p} \leq k,$$

then the stored energy function W_h given by (2.1) *is $W^{1,p}$-quasiconvex at $\mathbf{A}$ over the class $\mathcal{A}_{\mathbf{A},p}$.*

We now give the proof of Theorem 2.1.

Proof of Theorem 2.1. We first note that the first inequality in (2.7) together with the triangle inequality yields the second inequality. Let $p \in [n-1, n)$, $\mathbf{u} \in \mathcal{A}_{\mathbf{A},p}$, and define $\mathbf{u}^e$ by (1.52). Then, by (1.53),

$$\int_\Omega [\det\mathbf{A} - \det\nabla\mathbf{u}]\,d\mathbf{x} = m_{\mathbf{u}}(\overline{\Omega}).$$

Next, let $M \subset \overline{\Omega}$ be the support of $m_{\mathbf{u}}$. Then there is an $N \subset M$ with $m_{\mathbf{u}}(N) = 0$ such that

$$\lim_{r\to 0^+} \frac{m_{\mathbf{u}}(\overline{B(\mathbf{a},r)})}{r^n} = +\infty \quad \text{for every } \mathbf{a} \in M \setminus N, \tag{2.9}$$

since $m_{\mathbf{u}}$ is singular with respect to Lebesgue measure, see [17, Section 1.6]. Let $\mathbf{a} \in M \setminus N$. By (1.51),

$$(\mathrm{Det}\nabla\mathbf{u}^e)(B(\mathbf{a},t)) = \mathcal{L}^n(\mathrm{im}_T(\mathbf{u}^e, B(\mathbf{a},t))) \tag{2.10}$$

for a.e. $t > 0$, while Propositions 1.25, 1.36 and 1.28 imply that, for such t,

$$\begin{aligned}\mathcal{L}^n(\mathrm{im}_T(\mathbf{u}^e, B(\mathbf{a},t)))^{(n-1)/n} &\le \omega\mathcal{H}^{n-1}(\partial^*\mathrm{im}_T(\mathbf{u}^e, B(\mathbf{a},t)))\\ &\le \omega\mathcal{H}^{n-1}(\mathbf{u}^e(\partial B(\mathbf{a},t)))\\ &\le (n-1)^{(1-n)/2}\omega\int_{\partial B(\mathbf{a},t)}|\nabla\mathbf{u}^e|^{n-1}\,d\mathcal{H}^{n-1}.\end{aligned}\tag{2.11}$$

In view of (1.24) and the non-negativity of $m_{\mathbf{u}}$ and $\det\nabla\mathbf{u}^e$ we can combine (2.10) and (2.11) to conclude that

$$[m_{\mathbf{u}}(\overline{B(\mathbf{a},r)})]^{(n-1)/n} \le (n-1)^{(1-n)/2}\omega\int_{\partial B(\mathbf{a},t)}|\nabla\mathbf{u}^e|^{n-1}\,d\mathcal{H}^{n-1}$$

for almost every $r > 0$ and almost every $t > r$.

Fix $\varepsilon > 0$. Integrating the last inequality with respect to t over the interval $(r, (1+\varepsilon)r)$ we conclude that

$$\varepsilon r[m_{\mathbf{u}}(\overline{B(\mathbf{a},r)})]^{(n-1)/n} \le \kappa\int_{B(\mathbf{a},(1+\varepsilon)r)\setminus B(\mathbf{a},r)}|\nabla\mathbf{u}^e|^{n-1}\,d\mathbf{x},$$

where the constant κ is given by

$$\kappa := (n-1)^{(1-n)/2}\omega.$$

Dividing the previous relation by r^n, we get

$$\begin{aligned}\left[\frac{m_{\mathbf{u}}(\overline{B(\mathbf{a},r)})}{r^n}\right]^{(n-1)/n} &\le \frac{\kappa}{\varepsilon}\frac{1}{r^n}\int_{B_{(1+\varepsilon)r}\setminus B_r}|\nabla\mathbf{u}^e|^{n-1}\,d\mathbf{x},\\ &= \kappa\mathcal{L}^n(B_1)\frac{(1+\varepsilon)^n-1}{\varepsilon}\fint_{B_{(1+\varepsilon)r}\setminus B_r}|\nabla\mathbf{u}^e|^{n-1}\,d\mathbf{x},\end{aligned}$$

where we denote by $B_r := B(\mathbf{a},r)$. It then follows that

$$\left[\frac{m_{\mathbf{u}}(\overline{B(\mathbf{a},r)})}{r^n}\right]^{1/n} \le \left(\kappa\mathcal{L}^n(B_1)\frac{(1+\varepsilon)^n-1}{\varepsilon}\right)^{1/(n-1)}\|\nabla\mathbf{u}^e\|_{n-1},\tag{2.12}$$

where $\|\cdot\|_m$ denotes

$$\|\phi\|_m := \left(\fint_{B_{(1+\epsilon)r}\setminus B_r} |\phi(\mathbf{x})|^m \, d\mathbf{x}\right)^{1/m}, \quad m \geq 1.$$

Since $n-1 \leq p$, it follows from Hölder's inequality and the triangle inequality that

$$\begin{aligned}
\|\nabla \mathbf{u}^e\|_{n-1} &\leq \|\nabla \mathbf{u}^e\|_p \leq \||\nabla \mathbf{u}^e| - |\mathbf{A}|\|_p + \|\mathbf{A}\|_p = \\
&= \left(\fint_{B_{(1+\epsilon)r}\setminus B_r} \left||\nabla \mathbf{u}^e| - |\mathbf{A}|\right|^p dx\right)^{1/p} + \left(\fint_{B_{(1+\epsilon)r}\setminus B_r} |\mathbf{A}|^p \, d\mathbf{x}\right)^{1/p} \\
&= \left(\fint_{B_{(1+\epsilon)r}\setminus B_r} \left||\nabla \mathbf{u}^e| - |\mathbf{A}|\right|^p dx\right)^{1/p} + |\mathbf{A}| \\
&= \frac{n}{p}\left\{\frac{p}{n}\left[\left(\fint_{B_{(1+\epsilon)r}\setminus B_r} \left||\nabla \mathbf{u}^e| - |\mathbf{A}|\right|^p dx\right)^{1/n}\right]^{n/p}\right. \\
&\qquad\qquad \left. + \frac{n-p}{n}\left[\left(\frac{p}{n-p}|\mathbf{A}|\right)^{(n-p)/n}\right]^{n/(n-p)}\right\}.
\end{aligned} \tag{2.13}$$

We now use Young's inequality in the form

$$\frac{p}{n}c^{n/p} + \frac{n-p}{n}d^{n/(n-p)} \geq cd \qquad \text{for all } c, d > 0. \tag{2.14}$$

Let

$$d_0 := \left(\frac{p}{n-p}|\mathbf{A}|\right)^{(n-p)/n},$$

and let $c := c_0$ be the unique number such that equality holds in (2.14). Let

$$A := \left\{r \in (0,\infty) : \left(\fint_{B_{(1+\epsilon)r}\setminus B_r} \left||\nabla \mathbf{u}^e| - |\mathbf{A}|\right|^p dx\right)^{1/n} = c_0\right\}.$$

It follows from (2.9) and (2.12) that

$$\lim_{r\to 0^+} \fint_{B_{(1+\epsilon)r}\setminus B_r} \left||\nabla \mathbf{u}^e| - |\mathbf{A}|\right|^p d\mathbf{x} = +\infty. \tag{2.15}$$

Note also that

$$\lim_{r\to\infty} ⨍_{B_{(1+\varepsilon)r}\setminus B_r} \Big||\nabla \mathbf{u}^e| - |\mathbf{A}|\Big|^p \, d\mathbf{x} = 0. \tag{2.16}$$

From (2.15), (2.16) and the continuity of the mapping

$$r \mapsto ⨍_{B_{(1+\varepsilon)r}\setminus B_r} \Big||\nabla \mathbf{u}^e| - |\mathbf{A}|\Big|^p \, d\mathbf{x}$$

we deduce that $A \neq \emptyset$, A is bounded away from 0 and is a closed set. So, let $\rho_{\mathbf{a}} := \min A$, $\rho_{\mathbf{a}} > 0$. Thus, by definition, $\rho_{\mathbf{a}}$ is the least number r with the property that

$$\begin{aligned}\left(⨍_{B_{(1+\varepsilon)r}\setminus B_r} \Big||\nabla \mathbf{u}^e| - |\mathbf{A}|\Big|^p \, d\mathbf{x}\right)^{1/p} &+ |\mathbf{A}| = \\ = \frac{n}{p}\left(\frac{p}{n-p}\right)^{(n-p)/n} |\mathbf{A}|^{(n-p)/n} &\left(⨍_{B_{(1+\varepsilon)r}\setminus B_r} \Big||\nabla \mathbf{u}^e| - |\mathbf{A}|\Big|^p \, d\mathbf{x}\right)^{1/n}.\end{aligned} \tag{2.17}$$

Taking $r := \rho_{\mathbf{a}}$ in (2.12), it follows from (2.13) that

$$\left[\frac{m_{\mathbf{u}}(\overline{B(\mathbf{a},\rho_{\mathbf{a}})})}{\rho_{\mathbf{a}}^n}\right]^{1/n} \leq \kappa_\varepsilon |\mathbf{A}|^{(n-p)/n} \left(⨍_{B_{(1+\varepsilon)\rho_{\mathbf{a}}}\setminus B_{\rho_{\mathbf{a}}}} \Big||\nabla \mathbf{u}^e| - |\mathbf{A}|\Big|^p \, d\mathbf{x}\right)^{1/n}, \tag{2.18}$$

where

$$\kappa_\varepsilon := \left(\kappa \mathcal{L}^n(B_1)\frac{(1+\varepsilon)^n - 1}{\varepsilon}\right)^{1/(n-1)} \frac{n}{p}\left(\frac{p}{n-p}\right)^{(n-p)/n}.$$

Thus

$$\frac{m_{\mathbf{u}}(\overline{B(\mathbf{a},\rho_{\mathbf{a}})})}{\rho_{\mathbf{a}}^n} \leq \kappa_\varepsilon^n |\mathbf{A}|^{n-p} ⨍_{B_{(1+\varepsilon)\rho_{\mathbf{a}}}\setminus B_{\rho_{\mathbf{a}}}} \Big||\nabla \mathbf{u}^e| - |\mathbf{A}|\Big|^p \, d\mathbf{x}. \tag{2.19}$$

By the definition of $\rho_{\mathbf{a}}$ it follows that

$$⨍_{B_{(1+\varepsilon)r}\setminus B_r} \Big||\nabla \mathbf{u}^e| - |\mathbf{A}|\Big|^p \, d\mathbf{x} > c_0^n \qquad \text{for all } r < \rho_{\mathbf{a}}. \tag{2.20}$$

Let $\delta := 1/(1+\varepsilon)$. Then, by (2.20),

$$\begin{aligned}
&\int_{B_{\rho_{\mathbf{a}}}} \Big||\nabla \mathbf{u}^e| - |\mathbf{A}|\Big|^p \, d\mathbf{x} \\
&= \sum_{k=0}^{\infty} \mathcal{L}^n(B_{\delta^k \rho_{\mathbf{a}}} \setminus B_{\delta^{k+1}\rho_{\mathbf{a}}}) \fint_{B_{\delta^k \rho_{\mathbf{a}}} \setminus B_{\delta^{k+1}\rho_{\mathbf{a}}}} \Big||\nabla \mathbf{u}^e| - |\mathbf{A}|\Big|^p \, d\mathbf{x} \\
&> \sum_{k=0}^{\infty} \mathcal{L}^n(B_{\delta^k \rho_{\mathbf{a}}} \setminus B_{\delta^{k+1}\rho_{\mathbf{a}}}) c_0^n = \mathcal{L}^n(B_{\rho_{\mathbf{a}}}) c_0^n.
\end{aligned} \tag{2.21}$$

Hence

$$\fint_{B_{\rho_{\mathbf{a}}}} \Big||\nabla \mathbf{u}^e| - |\mathbf{A}|\Big|^p \, d\mathbf{x} > c_0^n = \fint_{B_{(1+\varepsilon)\rho_{\mathbf{a}}} \setminus B_{\rho_{\mathbf{a}}}} \Big||\nabla \mathbf{u}^e| - |\mathbf{A}|\Big|^p \, d\mathbf{x}. \tag{2.22}$$

It now follows from (2.19) that

$$\frac{m_{\mathbf{u}}(\overline{B(\mathbf{a}, \rho_{\mathbf{a}})})}{\rho_{\mathbf{a}}^n} < \kappa_\varepsilon^n |\mathbf{A}|^{n-p} \fint_{B_{\rho_{\mathbf{a}}}} \Big||\nabla \mathbf{u}^e| - |\mathbf{A}|\Big|^p \, d\mathbf{x},$$

so that

$$m_{\mathbf{u}}(\overline{B(\mathbf{a}, \rho_{\mathbf{a}})}) < \frac{\kappa_\varepsilon^n}{\mathcal{L}^n(B_1)} |\mathbf{A}|^{n-p} \int_{B_{\rho_{\mathbf{a}}}} \Big||\nabla \mathbf{u}^e| - |\mathbf{A}|\Big|^p \, d\mathbf{x}.$$

The same covering argument as in [35] yields the conclusion

$$m_{\mathbf{u}}(\overline{\Omega}) < \varpi_n \frac{\kappa_\varepsilon^n}{\mathcal{L}^n(B_1)} |\mathbf{A}|^{n-p} \int_\Omega \Big||\nabla \mathbf{u}^e| - |\mathbf{A}|\Big|^p \, d\mathbf{x}, \tag{2.23}$$

where ϖ_n is the constant in the Besicovitch Covering Theorem 1.21. Since (2.23) is true for all $\varepsilon > 0$, then, letting ε tend to 0 and using the fact that

$$\lim_{\varepsilon \to 0^+} \frac{(1+\varepsilon)^n - 1}{\varepsilon} = n,$$

we obtain that

$$m_{\mathbf{u}}(\overline{\Omega}) \leq \varpi_n (n-1)^{-n/2} \frac{n^n}{p^p (n-p)^{n-p}} |\mathbf{A}|^{n-p} \int_\Omega \Big||\nabla \mathbf{u}^e| - |\mathbf{A}|\Big|^p \, d\mathbf{x}.$$

This means, using (1.53), that

$$\int_{\Omega} [\det \mathbf{A} - \det \nabla \mathbf{u}]\, dx \le \mu |\mathbf{A}|^{n-p} \int_{\Omega} \Big| |\nabla \mathbf{u}^e| - |\mathbf{A}| \Big|^p dx,$$

where

$$\mu := \varpi_n (n-1)^{-n/2} \frac{n^n}{p^p (n-p)^{n-p}}.$$

An examination of the above proof shows that, if the singular measure $m_{\mathbf{u}}$ consists of a finite number N of Dirac measures (as in (1.25)), where $N \le \varpi_n$, the above value of μ can be improved to

$$N (n-1)^{-n/2} \frac{n^n}{p^p (n-p)^{n-p}}.$$

This completes the proof of Theorem 2.1. □

2.2 Sufficient condition for $W^{1,p}$-quasiconvexity of W_h, depending only on the determinant of the matrix A

We now derive an inequality, see (2.25) below, which is qualitatively better than (2.7), although valid for the more restrictive class of deformations producing a single hole anywhere in the material. Let $p \in (n-1, n)$, $\mathbf{D} := \operatorname{diag}(\lambda_1, \lambda_2, ..., \lambda_n)$ be a diagonal matrix, where $\lambda_1, ..., \lambda_n > 0$.

Theorem 2.4. *There exists a positive constant C_1 such that, for all mappings $\mathbf{u} \in \mathcal{A}^*_{\mathbf{D},p}$, the following holds*

$$\begin{aligned} &\int_{\Omega} [\det \mathbf{D} - \det \nabla \mathbf{u}(\mathbf{x})]\, dx \\ &\quad \le C_1 (\det \mathbf{D})^{\frac{n-p}{n}} \left(\int_{\Omega} |\nabla u_1 - \lambda_1 \mathbf{e}_1|^p\, dx \right)^{1/n} \cdots \left(\int_{\Omega} |\nabla u_n - \lambda_n \mathbf{e}_n|^p\, dx \right)^{1/n}. \end{aligned} \tag{2.24}$$

It then follows that there exists a positive constant C_2 such that, for all $\mathbf{u} \in \mathcal{A}^*_{\mathbf{D},p}$,

$$\int_{\Omega} [\det \mathbf{D} - \det \nabla \mathbf{u}(\mathbf{x})]\, dx \le C_2 (\det \mathbf{D})^{\frac{n-p}{n}} \int_{\Omega} |\nabla \mathbf{u}(\mathbf{x}) - \mathbf{D}|^p\, dx. \tag{2.25}$$

Theorem 2.4 leads to the following $W^{1,p}$-quasiconvexity result.

Theorem 2.5. *Let $\bar{k} := K/C_2$, where K is as in* (2.8) *and C_2 is as in Theorem 2.4. If $\mathbf{A} \in M_+^{3\times 3}$ satisfies*

$$h'(\det \mathbf{A})(\det \mathbf{A})^{(3-p)/3} \leq \bar{k}, \tag{2.26}$$

then the stored energy function W_h given by (2.1) *is $W^{1,p}$-quasiconvex at $\mathbf{A}$ over the class $\mathcal{A}^*_{\mathbf{A},p}$.*

Proof of Theorem 2.5. As noted in Proposition 1.50, the $W^{1,p}$-quasiconvexity at a matrix $\mathbf{A}$ depends only on the principal stretches of $\mathbf{A}$. As the condition (2.26) also depends only on the principal stretches of $\mathbf{A}$, it follows that it suffices for the proof of Theorem 2.5 to consider only the case of diagonal matrices. In this case, the required result is an immediate consequence of Theorem 2.4, using (2.8) and the convexity of h. $\square$

The key to the proof of Theorem 2.4 is the following result.

Proposition 2.6. *Let $\mathbf{u}$, $\mathbf{a}$ and $N_\mathbf{a}$ be as in Proposition 1.37. Then*

$$t \mapsto \sup_{\mathbf{x},\mathbf{y}\in\partial B(\mathbf{a},t)} |u_i(\mathbf{x}) - u_i(\mathbf{y})| \qquad \textit{for } i = 1, ..., n,$$

is an increasing function on $(0, r_\mathbf{a}) \setminus N_\mathbf{a}$.

Proof. Let $t \in (0, r_\mathbf{a}) \setminus N_\mathbf{a}$. As noted in Theorem 1.16, the value of the degree of $\mathbf{u}$ is 0 in the unique unbounded component of $\mathbb{R}^n \setminus \mathbf{u}(\partial B(\mathbf{a}, t))$. For $i = 1, ..., n$, let

$$c_i = \min\{u_i(\mathbf{x}) : \mathbf{x} \in \partial B(\mathbf{a}, t)\}, \qquad d_i = \max\{u_i(\mathbf{x}) : \mathbf{x} \in \partial B(\mathbf{a}, t)\}. \tag{2.27}$$

Then $\mathbb{R}^n \setminus [c_1, d_1] \times [c_2, d_2] \times ... \times [c_n, d_n]$ is contained in the unbounded connected component of $\mathbb{R}^n \setminus \mathbf{u}(\partial B(\mathbf{a}, t))$. We deduce that

$$\operatorname{im}_T(\mathbf{u}, B(\mathbf{a}, t)) \subset [c_1, d_1] \times [c_2, d_2] \times ... \times [c_n, d_n]. \tag{2.28}$$

Let now $s \in (0, r_\mathbf{a}) \setminus N_\mathbf{a}$ with $s < t$. We deduce from (1.48) and (2.28) that

$$\mathbf{u}(\partial B(\mathbf{a}, s)) \subset [c_1, d_1] \times [c_2, d_2] \times ... \times [c_n, d_n].$$

This shows that, for all $i = 1, ..., n$,

$$\sup_{\mathbf{x},\mathbf{y}\in\partial B(\mathbf{a},s)} |u_i(\mathbf{x}) - u_i(\mathbf{y})| \leq \sup_{\mathbf{x},\mathbf{y}\in\partial B(\mathbf{a},t)} |u_i(\mathbf{x}) - u_i(\mathbf{y})|. \tag{2.29}$$

Since s, t with $s < t$ were arbitrary in $(0, r_{\mathbf{a}}) \setminus N_{\mathbf{a}}$, the required result follows. $\square$

Proof of Theorem 2.4. Note first that the inequality (2.25) is an immediate consequence of (2.24), since

$$\begin{aligned}
&\int_\Omega |\nabla \mathbf{u} - \mathbf{D}|^p \, d\mathbf{x} = \int_\Omega \left(|\nabla u_1 - \lambda_1 \mathbf{e}_1|^2 + ... + |\nabla u_n - \lambda_n \mathbf{e}_n|^2 \right)^{p/2} d\mathbf{x} \\
&\geq M \left(\int_\Omega |\nabla u_1 - \lambda_1 \mathbf{e}_1|^p \, d\mathbf{x} + ... + \int_\Omega |\nabla u_n - \lambda_n \mathbf{e}_n|^p \, d\mathbf{x} \right) \\
&\geq Mn \left(\int_\Omega |\nabla u_1 - \lambda_1 \mathbf{e}_1|^p \, d\mathbf{x} \right)^{1/n} ... \left(\int_\Omega |\nabla u_n - \lambda_n \mathbf{e}_n|^p \, d\mathbf{x} \right)^{1/n}
\end{aligned}$$

(Here M is such that

$$(a_1^2 + ... + a_n^2)^{1/2} \geq M^{1/p} (|a_1|^p + ... + |a_n|^p)^{1/p}, \qquad \text{for all } (a_1, ..., a_n) \in \mathbb{R}^n,$$

since all norms in a finite-dimensional space are equivalent.)

Let $\mathbf{u}$ be a deformation in $\mathcal{A}^*_{\mathbf{D},p}$. Let $\mathbf{a} \in \Omega$ be such that $\mathbf{u} \in \mathcal{A}_{\mathbf{D},p}(\mathbf{a})$, and let $V \geq 0$ be such that $m_{\mathbf{u}} = V\delta_{\mathbf{a}}$. For simplicity, we use the notation $\mathbf{u}$ instead of $\mathbf{u}^e$ for the homogeneous extension of $\mathbf{u}$, and we denote by B_t the ball $B(\mathbf{a}, t)$, for any $t > 0$. Since $\mathbf{u}$ satisfies (INV), it follows from (1.50), (1.51) and (2.28) that, for almost every $t \in (0, 1)$,

$$\begin{aligned}
V = m_{\mathbf{u}}(B_t) &\leq \mathcal{L}^n(\mathrm{im}_T(\mathbf{u}, B_t)) \\
&\leq \sup_{\mathbf{x},\mathbf{y}\in\partial B_t} |u_1(\mathbf{x}) - u_1(\mathbf{y})| \cdot ... \cdot \sup_{\mathbf{x},\mathbf{y}\in\partial B_t} |u_n(\mathbf{x}) - u_n(\mathbf{y})|.
\end{aligned} \tag{2.30}$$

The crucial fact in proving (2.24) is the observation that actually the following inequality, which is much stronger than (2.30), is satisfied: for almost every $t_1, t_2, ..., t_n \in (0, \infty)$,

$$V \leq \sup_{\mathbf{x},\mathbf{y}\in\partial B_{t_1}} |u_1(\mathbf{x}) - u_1(\mathbf{y})| \cdot ... \cdot \sup_{\mathbf{x},\mathbf{y}\in\partial B_{t_n}} |u_n(\mathbf{x}) - u_n(\mathbf{y})|. \tag{2.31}$$

Indeed, (2.31) follows by applying (2.30) for some $t \leq \min\{t_1, .., t_n\}$, and taking into account Proposition 2.6.

Let $r_1, \ldots, r_n \in (0, \infty)$. Using the above inequality, it follows immediately from the the Sobolev Imbedding Theorem (Proposition 1.27) applied to the components of $\mathbf{u}$ that there exists a constant $C > 0$ such that, for almost every $t_i \in (0, r_i)$, $i = 1, \ldots, n$,

$$\begin{aligned} V^p &\leq \sup_{\mathbf{x},\mathbf{y} \in \partial B_{t_1}} |u_1(\mathbf{x}) - u_1(\mathbf{y})|^p \cdot \ldots \cdot \sup_{\mathbf{x},\mathbf{y} \in \partial B_{t_n}} |u_n(\mathbf{x}) - u_n(\mathbf{y})|^p \\ &\leq C t_1^{p-n+1} \int_{\partial B_{t_1}} |\nabla u_1|^p \, d\mathcal{H}^{n-1} \ldots t_n^{p-n+1} \int_{\partial B_{t_n}} |\nabla u_n|^p \, d\mathcal{H}^{n-1} \\ &\leq C r_1^{p-n+1} \int_{\partial B_{t_1}} |\nabla u_1|^p \, d\mathcal{H}^{n-1} \ldots r_n^{p-n+1} \int_{\partial B_{t_n}} |\nabla u_n|^p \, d\mathcal{H}^{n-1}. \end{aligned} \tag{2.32}$$

Integrating now with respect to $t_1 \in (0, r_1)$, $t_2 \in (0, r_2)$, ... $t_n \in (0, r_n)$, we get that there exist positive constants $C_0, \widetilde{C}_0, \widehat{C}_0$ such that

$$\begin{aligned} V^p &\leq C_0 \left(r_1^{p-n} \int_{B_{r_1}} |\nabla u_1|^p \, d\mathbf{x} \right) \ldots \left(r_n^{p-n} \int_{B_{r_n}} |\nabla u_n|^p \, d\mathbf{x} \right) \\ &\leq \widetilde{C}_0 \left(r_1^p \fint_{B_{r_1}} |\nabla u_1|^p \, d\mathbf{x} \right) \ldots \left(r_n^p \fint_{B_{r_n}} |\nabla u_n|^p \, d\mathbf{x} \right) \\ &\leq \widehat{C}_0 r_1^p \left(\fint_{B_{r_1}} |\nabla u_1 - \lambda_1 \mathbf{e}_1|^p \, d\mathbf{x} + |\lambda_1 \mathbf{e}_1|^p \right) \times \ldots \\ &\qquad \times r_n^p \left(\fint_{B_{r_n}} |\nabla u_n - \lambda_n \mathbf{e}_n|^p \, d\mathbf{x} + |\lambda_n \mathbf{e}_n|^p \right). \end{aligned} \tag{2.33}$$

It is immediate from above that

$$\lim_{r_i \to 0} \fint_{B_{r_i}} |\nabla u_i - \lambda_i \mathbf{e}_i|^p \, d\mathbf{x} = +\infty \quad \text{for all } i = 1, \ldots, n. \tag{2.34}$$

Note also that, in view of the compact support of the integrand,

$$\lim_{r_i \to \infty} \fint_{B_{r_i}} |\nabla u_i - \lambda_i \mathbf{e}_i|^p \, d\mathbf{x} = 0 \quad \text{for all } i = 1, \ldots, n. \tag{2.35}$$

Using the continuity of the functions

$$r \mapsto \fint_{B_r} |\nabla u_i - \lambda_i \mathbf{e}_i|^p \, d\mathbf{x}, \qquad r \in (0,\infty),\ i = 1, ..., n,$$

it follows from (2.34) and (2.35) that there exist ρ_i, $i = 1, ..., n$, with the property that

$$\fint_{B_{\rho_i}} |\nabla u_i - \lambda_i \mathbf{e}_i|^p \, d\mathbf{x} = |\lambda_i \mathbf{e}_i|^p,$$

so that

$$\fint_{B_{\rho_i}} |\nabla u_i - \lambda_i \mathbf{e}_i|^p \, d\mathbf{x} + |\lambda_i \mathbf{e}_i|^p = 2 \left(\fint_{B_{\rho_i}} |\nabla u_i - \lambda_i \mathbf{e}_i|^p \, d\mathbf{x} \right)^{p/n} (\lambda_i |\mathbf{e}_i|)^{p(1-\frac{p}{n})}.$$

We therefore obtain that there exists $\widetilde{C} > 0$ such that

$$\begin{aligned}
\fint_{B_{\rho_i}} |\nabla u_i - \lambda_i \mathbf{e}_i|^p \, d\mathbf{x} + |\lambda_i \mathbf{e}_i|^p &= 2 \left(\fint_{B_{\rho_i}} |\nabla u_i - \lambda_i \mathbf{e}_i|^p \, d\mathbf{x} \right)^{p/n} (\lambda_i |\mathbf{e}_i|)^{p(1-\frac{p}{n})} \\
&= \widetilde{C} \left(\frac{1}{\rho_i^n} \int_{B_{\rho_i}} |\nabla u_i - \lambda_i \mathbf{e}_i|^p \, d\mathbf{x} \right)^{p/n} \lambda_i^{p(1-\frac{p}{n})} \\
&= \widetilde{C} \frac{\lambda_i^{p(1-\frac{p}{n})}}{\rho_i^p} \left(\int_{B_{\rho_i}} |\nabla u_i - \lambda_i \mathbf{e}_i|^p \, d\mathbf{x} \right)^{p/n} \\
&\leq \widetilde{C} \frac{\lambda_i^{p(1-\frac{p}{n})}}{\rho_i^p} \left(\int_{\Omega} |\nabla u_i - \lambda_i \mathbf{e}_i|^p \, d\mathbf{x} \right)^{p/n}.
\end{aligned}$$

Using this in (2.33), we deduce that there exists a constant $C_1 > 0$ such that

$$V^p \leq C_1^p (\lambda_1 \lambda_2 ... \lambda_n)^{p(1-\frac{p}{n})} \left(\int_{\Omega} |\nabla u_1 - \lambda_1 \mathbf{e}_1|^p \, d\mathbf{x} \right)^{p/n} ... \left(\int_{\Omega} |\nabla u_n - \lambda_n \mathbf{e}_n|^p \, d\mathbf{x} \right)^{p/n},$$

and hence the desired inequality

$$V \leq C_1 (\lambda_1 \lambda_2 ... \lambda_n)^{\frac{n-p}{n}} \left(\int_{\Omega} |\nabla u_1 - \lambda_1 \mathbf{e}_1|^p \, d\mathbf{x} \right)^{1/n} ... \left(\int_{\Omega} |\nabla u_n - \lambda_n \mathbf{e}_n|^p \, d\mathbf{x} \right)^{1/n}.$$

This completes the proof of Theorem 2.4. □

Chapter 3

Critical Values for $W^{1,p}$-quasiconvexity

In this chapter we study the $W^{1,p}$-quasiconvexity at $\lambda\mathbf{I}$ in the class of deformations $\mathcal{A}^*_{\lambda\mathbf{I},p}$ opening a single hole anywhere in the material of the stored energy function

$$W_h(\mathbf{F}) = |\mathbf{F}|^p + h(\det \mathbf{F}) \qquad \text{for all } \mathbf{F} \in M^{3\times 3}_+, \tag{3.1}$$

where $p \in [2, 3)$ and the function h satisfies (1.30). Our approach is based on investigating first the related model energy function

$$W_\alpha(\mathbf{F}) = |\mathbf{F}|^p + \alpha \det \mathbf{F} \qquad \text{for all } \mathbf{F} \in M^{3\times 3}_+. \tag{3.2}$$

The motivation for our study comes from the paper [41] of Sivaloganathan, where he considered the case when $p = 2$ in (3.1) and (3.2). He proved that the stored energy function given by $W_\alpha(\mathbf{F}) = |\mathbf{F}|^2 + \alpha \det \mathbf{F}$ is $W^{1,p}$-quasiconvex at $\lambda\mathbf{I}$ over the class $\mathcal{A}^*_{\lambda\mathbf{I},2}$ if and only if $\lambda\alpha \leq 8/3$.

Here we consider the general case $p \in [2, 3)$ in (3.1) and (3.2). We completely characterize the $W^{1,p}$-quasiconvexity over the class $\mathcal{A}^*_{\lambda\mathbf{I},p}$ of the model energy function given by (3.2).

Theorem 3.1. *For any $p \in [2, 3)$, the model energy function W_α given by* (3.2) *is $W^{1,p}$-quasiconvex at $\lambda\mathbf{I}$ over the class $\mathcal{A}^*_{\lambda\mathbf{I},p}$ if and only if $\alpha\lambda^{3-p} \leq \Upsilon_p$, where*

$$\Upsilon_p := 2^{p/2} \frac{p}{3-p} \exp\left\{(p-3) \int_0^1 \frac{(p-1)y^2 + 2}{(y^2+2)[(p-1)y+2]}\, dy\right\}. \tag{3.3}$$

Remark 3.2. The integral in (3.3) can be evaluated to yield

$$\Upsilon_p = 2^{p/2}\frac{p}{3-p}\left[(p+1)^{\frac{(p+1)(p-3)}{p^2-2p+3}} \times \frac{3^{\frac{(p-1)(p-2)(p-3)}{2(p-1)^2+4}}}{2^{\frac{(p^2-p+4)(p-3)}{2(p-1)^2+4}}} \times e^{\frac{-\sqrt{2}(p-2)(p-3)}{(p-1)^2+2}\arctan\frac{1}{\sqrt{2}}}\right]. \quad (3.4)$$

Remark 3.3. For $p=2$, the value $\Upsilon_2 = 8/3$ was first obtained by Sivaloganathan [41].

An immediate consequence of Theorem 3.1 is the following $W^{1,p}$-quasiconvexity result concerning the stored energy function given by (3.1).

Theorem 3.4. *For any $p \in [2,3)$, if $\lambda^{3-p}h'(\lambda^3) \leq \Upsilon_p$ then the stored energy function W_h given by* (3.1) *is $W^{1,p}$-quasiconvex at $\lambda\mathbf{I}$ over the class $\mathcal{A}^*_{\lambda\mathbf{I},p}$.*

The proof of Theorem 3.4 is similar to that in [41] for $p=2$. Namely, it follows from Theorem 3.1 and the convexity of h that, for W_h given by (3.1),

$$\begin{aligned}
E(\mathbf{u}) - E(\mathbf{u}^{\text{hom}}_{\lambda\mathbf{I}}) &\geq \int_\Omega |\nabla\mathbf{u}|^p - |\lambda\mathbf{I}|^p + h'(\lambda^3)(\det\nabla\mathbf{u} - \det\lambda\mathbf{I})\,d\mathbf{x} \\
&= \int_\Omega |\nabla\mathbf{u}|^p + h'(\lambda^3)\det\nabla\mathbf{u}\,d\mathbf{x} - \int_\Omega |\lambda\mathbf{I}|^p + h'(\lambda^3)\det\lambda\mathbf{I}\,d\mathbf{x} \\
&\geq 0, \quad \text{if } \lambda^{3-p}h'(\lambda^3) \leq \Upsilon_p.
\end{aligned}$$

Hence the condition $\lambda^{3-p}h'(\lambda^3) \leq \Upsilon_p$ implies that $E(\mathbf{u}^{\text{hom}}_{\lambda\mathbf{I}}) \leq E(\mathbf{u})$ for all maps $\mathbf{u}$ opening a single hole in the material.

3.1 Outline of the proof of Theorem 3.1

Since the $W^{1,p}$-quasiconvexity is independent of the domain, there is no loss of generality in studying deformations of the unit ball.

The model energy function W_α given by (3.2) is $W^{1,p}$-quasiconvex at $\lambda\mathbf{I}$ over the class $\mathcal{A}^*_{\lambda\mathbf{I},p}$ if and only if, for all $\mathbf{v} \in \mathcal{A}^*_{\lambda\mathbf{I},p}$,

$$\alpha\int_\Omega \det\nabla\mathbf{v}^{\text{hom}}_{\lambda\mathbf{I}} - \det\nabla\mathbf{v}\,d\mathbf{x} \leq \int_\Omega |\nabla\mathbf{v}|^p - |\nabla\mathbf{v}^{\text{hom}}_{\lambda\mathbf{I}}|^p\,d\mathbf{x}. \quad (3.5)$$

Note that (3.5) is automatically satisfied for functions $\mathbf{v}$ such that $m_\mathbf{v}(\overline{\Omega}) = 0$, so it suffices to restrict attention only to those with $m_\mathbf{v}(\overline{\Omega}) \neq 0$. Upon writing

$\mathbf{v} = \lambda\mathbf{u}$ and $\mathbf{v}_{\lambda\mathbf{I}}^{\text{hom}} = \lambda\mathbf{u}_{\mathbf{I}}^{\text{hom}}$, where $\mathbf{u} \in \mathcal{A}_{\mathbf{I},p}^{*}$, we get that (3.5) holds exactly when λ and α satisfy

$$\lambda^{3-p}\alpha \leq \Upsilon_p, \tag{3.6}$$

where

$$\Upsilon_p := \inf\left\{ \frac{\int_\Omega |\nabla\mathbf{u}|^p - |\nabla\mathbf{u}_{\mathbf{I}}^{\text{hom}}|^p \, d\mathbf{x}}{\int_\Omega \det\nabla\mathbf{u}_{\mathbf{I}}^{\text{hom}} - \det\nabla\mathbf{u}\, d\mathbf{x}} : \mathbf{u} \in \mathcal{A}_{\mathbf{I},p}^{*},\, m_{\mathbf{u}}(\overline{\Omega}) \neq 0 \right\}. \tag{3.7}$$

However, Lemma 1.49 shows that W_α is $W^{1,p}$-quasiconvex over the class $\mathcal{A}_{\lambda\mathbf{I},p}^{*}$ if and only if it is so over the class $\mathcal{A}_{\lambda\mathbf{I},p}(\mathbf{0})$. We deduce that Υ_p, originally defined in (3.7), satisfies also

$$\Upsilon_p = \inf\left\{ \frac{\int_\Omega |\nabla\mathbf{u}|^p - |\nabla\mathbf{u}_{\mathbf{I}}^{\text{hom}}|^p \, d\mathbf{x}}{\int_\Omega \det\nabla\mathbf{u}_{\mathbf{I}}^{\text{hom}} - \det\nabla\mathbf{u}\, d\mathbf{x}} : \mathbf{u} \in \mathcal{A}_{\mathbf{I},p}(\mathbf{0}),\, m_{\mathbf{u}}(\overline{\Omega}) \neq 0 \right\}. \tag{3.8}$$

For each $V \in (0, 4\pi/3]$, let

$$\mathcal{C}_V := \left\{ \mathbf{u} \in \mathcal{A}_{\mathbf{I},p}(\mathbf{0}) : \, m_{\mathbf{u}}(\overline{\Omega}) = V \right\}. \tag{3.9}$$

It is clear that

$$\Upsilon_p = \inf_{V\in(0,4\pi/3]} \inf_{\mathbf{u}\in\mathcal{C}_V} \frac{\int_\Omega |\nabla\mathbf{u}|^p - |\nabla\mathbf{u}_{\mathbf{I}}^{\text{hom}}|^p \, d\mathbf{x}}{\int_\Omega \det\nabla\mathbf{u}_{\mathbf{I}}^{\text{hom}} - \det\nabla\mathbf{u}\, d\mathbf{x}}. \tag{3.10}$$

Next note that if we consider $0 < V_2 < V_1$ and $\mathbf{u}_1 \in \mathcal{C}_{V_1}$, and if $\alpha \in (0,1)$ is such that $V_2 = \alpha^3 V_1$, then the mapping

$$\mathbf{u}_2(\mathbf{x}) = \begin{cases} \alpha\, \mathbf{u}_1(\mathbf{x}/\alpha), & \text{for } |\mathbf{x}| \in [0,\alpha], \\ \mathbf{x}, & \text{for } |\mathbf{x}| \in [\alpha, 1], \end{cases} \tag{3.11}$$

satisfies the following: $\mathbf{u}_2 \in \mathcal{C}_{V_2}$ and

$$\frac{\int_\Omega |\nabla\mathbf{u}_2|^p - |\nabla\mathbf{u}_{\mathbf{I}}^{\text{hom}}|^p \, d\mathbf{x}}{\int_\Omega \det\nabla\mathbf{u}_{\mathbf{I}}^{\text{hom}} - \det\nabla\mathbf{u}_2\, d\mathbf{x}} = \frac{\int_\Omega |\nabla\mathbf{u}_1|^p - |\nabla\mathbf{u}_{\mathbf{I}}^{\text{hom}}|^p \, d\mathbf{x}}{\int_\Omega \det\nabla\mathbf{u}_{\mathbf{I}}^{\text{hom}} - \det\nabla\mathbf{u}_1\, d\mathbf{x}}. \tag{3.12}$$

This shows that the function

$$V \mapsto \inf_{\mathbf{u}\in\mathcal{C}_V} \frac{\int_\Omega |\nabla \mathbf{u}|^p - |\nabla \mathbf{u}_\mathbf{I}^{\mathrm{hom}}|^p \, d\mathbf{x}}{\int_\Omega \det \nabla \mathbf{u}_\mathbf{I}^{\mathrm{hom}} - \det \nabla \mathbf{u} \, d\mathbf{x}}$$

is increasing, so that, from (3.10),

$$\begin{aligned}
\Upsilon_p &= \lim_{V \searrow 0} \inf_{\mathbf{u}\in\mathcal{C}_V} \frac{\int_\Omega |\nabla \mathbf{u}|^p - |\nabla \mathbf{u}_\mathbf{I}^{\mathrm{hom}}|^p \, d\mathbf{x}}{\int_\Omega \det \nabla \mathbf{u}_\mathbf{I}^{\mathrm{hom}} - \det \nabla \mathbf{u} \, d\mathbf{x}} \\
&= \lim_{V \searrow 0} \frac{\inf_{\mathbf{u}\in\mathcal{C}_V} \int_\Omega |\nabla \mathbf{u}|^p \, d\mathbf{x} - \int_\Omega |\nabla \mathbf{u}_\mathbf{I}^{\mathrm{hom}}|^p \, d\mathbf{x}}{V}.
\end{aligned} \tag{3.13}$$

The key fact for calculating the value of Υ_p from the above formula is that

$$\inf_{\mathbf{u}\in\mathcal{C}_V} \int_\Omega |\nabla \mathbf{u}|^p \, d\mathbf{x}$$

can be determined explicitly. For each $V \in (0, 4\pi/3]$, let $\beta \in (0,1]$ be such that $V = 4\pi\beta^3/3$. In Section 3.2 we show that there exists a function $r_\beta : [0,1] \to \mathbb{R}$ of class C^1 on $[0,1]$, which equals β on some interval $[0,\varepsilon]$ and satisfies the radial Euler-Lagrange equation for the functional $\mathbf{F} \mapsto |\mathbf{F}|^p$ on the interval $[\varepsilon, 1]$. In Section 3.3 we show, by means of an isoperimetric estimate, that the radial mapping $\mathbf{u}_\beta(\mathbf{x}) = r_\beta(|\mathbf{x}|)\frac{\mathbf{x}}{|\mathbf{x}|}$, satisfies

$$\inf\left\{\int_\Omega |\nabla \mathbf{u}|^p \, d\mathbf{x} : \, \mathbf{u} \in \mathcal{C}_V\right\} = \int_\Omega |\nabla \mathbf{u}_\beta|^p \, d\mathbf{x}. \tag{3.14}$$

Using this fact in (3.13), we deduce that

$$\Upsilon_p = \lim_{\beta \searrow 0} \frac{\int_\Omega |\nabla \mathbf{u}_\beta|^p - |\nabla \mathbf{u}_\mathbf{I}^{\mathrm{hom}}|^p \, d\mathbf{x}}{\int_\Omega \det \nabla \mathbf{u}_\mathbf{I}^{\mathrm{hom}} - \det \nabla \mathbf{u}_\beta \, d\mathbf{x}}. \tag{3.15}$$

From this formula, the value of Υ_p is shown to be equal to that in (3.3) by making use of certain conservation laws satisfied by r_β on the interval $[\varepsilon, 1]$.

3.2 The existence of $\mathbf{u}_\beta$

This section is devoted to the existence of a radial function $\mathbf{u}_\beta$ with some special properties.

Let us first recall some properties of radial deformations. Let $\mathbf{u}$ be of the form $\mathbf{u}(\mathbf{x}) = r(|\mathbf{x}|)\frac{\mathbf{x}}{|\mathbf{x}|}$, with $r \in A$, where

$$A := \left\{ r \in W^{1,1}(0,1) : r(1) = 1,\, r(0) > 0,\, r'(R) \geq 0 \text{ a.e. } R \in (0,1) \right\}. \tag{3.16}$$

Then

$$\int_\Omega \det \nabla \mathbf{u}\, d\mathbf{x} = 4\pi \int_0^1 r' r^2\, dR = 4\pi \left(\frac{1}{3} - \frac{r^3(0)}{3} \right).$$

If W and Φ are related by

$$W(\mathbf{F}) = \Phi(v_1, v_2, v_3), \tag{3.17}$$

where v_1, v_2, v_3 are the principal stretches of $\mathbf{F}$, then for any radial mapping $\mathbf{u}$ with $r \in A$,

$$E(\mathbf{u}) = \int_\Omega W(\nabla \mathbf{u})\, d\mathbf{x} = 4\pi \int_0^1 R^2 \Phi\left(r', \frac{r}{R}, \frac{r}{R}\right) dR =: 4\pi I(r).$$

An equivalent form of the radial Euler-Lagrange equation (1.62) is

$$r''(R) = \frac{2}{R} \left(\frac{\Phi_2 - \Phi_1 - \left(r'(R) - \frac{r(R)}{R}\right)\Phi_{12}}{\Phi_{11}} \right), \tag{3.18}$$

where the partial derivatives of Φ are evaluated at $\left(r'(R), \frac{r(R)}{R}, \frac{r(R)}{R}\right)$.

Let $W(\mathbf{F}) = |\mathbf{F}|^p$ for all $\mathbf{F} \in M^{3\times 3}_+$. Then the function Φ in (3.17) is given by

$$\Phi(v_1, v_2, v_3) = (v_1^2 + v_2^2 + v_3^2)^{p/2}, \tag{3.19}$$

and hence I is given by

$$I(r) := \int_0^1 R^2 \left[(r'(R))^2 + 2\left(\frac{r(R)}{R}\right)^2 \right]^{p/2} dR. \tag{3.20}$$

The Euler-Lagrange equation for I is

$$r''(R) = \frac{2}{R}\left[\frac{r(R)}{R} - r'(R)\right]\left(\frac{(r'(R))^2 + 2\left(\frac{r(R)}{R}\right)^2 + (p-2)\frac{r(R)r'(R)}{R}}{(r'(R))^2 + 2\left(\frac{r(R)}{R}\right)^2 + (p-2)(r'(R))^2}\right). \quad (3.21)$$

We shall prove that for any $\beta \in (0,1)$ there exists a function r_β, which is obtained by smoothly piecing together the constant function β on an interval $[0,\varepsilon]$ and a solution to the Euler-Lagrange equation (3.21) on $[\varepsilon, 1]$, for some $\varepsilon \in (0,1)$.

Theorem 3.5. *For every $\tilde{\beta} \in (0,1)$ there exist $\tilde{\varepsilon} \in (0,1)$ and a solution $\tilde{r}$ of the Euler-Lagrange equation* (3.21) *on the interval $[\tilde{\varepsilon}, 1]$ which satisfies $\tilde{r}(\tilde{\varepsilon}) = \tilde{\beta}$ and $\tilde{r}'(\tilde{\varepsilon}) = 0$.*

An important observation which will be used in the proof of Theorem 3.5 is that solutions of (3.21) satisfy the following conservation law on any interval where they exist.

Proposition 3.6. *Let r be a C^2 solution of* (3.21) *on an open interval J. Then there exists a constant $\kappa \in \mathbb{R}$ such that on J*

$$R^3\left[\Phi\left(r', \frac{r}{R}, \frac{r}{R}\right) - r'\Phi_1\left(r', \frac{r}{R}, \frac{r}{R}\right) + \gamma\frac{r}{R}\Phi_1\left(r', \frac{r}{R}, \frac{r}{R}\right)\right] = \kappa, \quad (3.22)$$

where Φ is given by (3.19) *and $\gamma := 1 - \frac{3}{p}$.*

Proof of Proposition 3.6. First notice that the variational integrand

$$f(R, r, z) = R^2\Phi\left(z, \frac{r}{R}, \frac{r}{R}\right), \quad (3.23)$$

with Φ given by (3.19), satisfies the following homogeneity condition

$$af(aR, a^\gamma r, a^{\gamma-1}z) = f(R, r, z) \quad \text{for all } a > 0, \quad (3.24)$$

where $\gamma = 1 - 3/p$. This invariance of the variational integrand ensures, by a theorem of Noether, that the first order expression

$$\chi(R, r, z) := R[f - zf_z](R, r, z) + \gamma r f_z(R, r, z)$$

is a first integral of the Euler-Lagrange equation, i.e

$$R[f - r'f_z](R, r, r') + \gamma r f_z(R, r, r') = \text{ constant },\tag{3.25}$$

for every solution r of the Euler-Lagrange equations for f. This immediately leads to (3.22).

Alternatively, the validity of the conservation law (3.25) can be checked directly. Indeed, taking $\rho(a) := af(aR, a^\gamma r, a^{\gamma-1}z)$, (3.24) ensures that ρ is a constant function, so that $\rho'(1) = 0$, and this gives a relation between the partial derivatives of f. Taking this relation into account, it is immediate that if r satisfies the Euler-Lagrange equation, then

$$\frac{d}{dR}\chi(R, r(R), r'(R)) = 0,$$

which proves (3.25). □

Let $\theta \in (0, 1)$, and consider equation (3.21) with initial conditions

$$r(1) = 1, \;\; r'(1) = \theta.\tag{3.26}$$

This problem is a second order ODE of the form

$$r''(R) = g(R, r(R), r'(R)),$$

where g is a smooth function on $(0, \infty) \times (\mathbb{R}^2 \setminus \{0\})$. The classical Peano-Picard Theorem ensures the existence of a unique solution $r : (j(\theta), 1] \to \mathbb{R}$, which is non-continuable to the left. By Proposition 3.6, for every $R \in (j(\theta), 1]$,

$$\begin{aligned} R^3\left[(r'(R))^2 + 2\left(\frac{r}{R}\right)^2\right]^{\frac{p}{2}-1}\left[(1-p)(r'(R))^2 + 2\left(\frac{r}{R}\right)^2 + (p-3)r'(R)\frac{r(R)}{R}\right] \\ = (\theta^2+2)^{\frac{p}{2}-1}[(1-p)\theta^2 + 2 + (p-3)\theta]. \end{aligned}\tag{3.27}$$

Lemma 3.7. *Let Φ be given by* (3.19). *Let $r : (j(\theta), 1] \to \mathbb{R}$ be a solution to* (3.21) *and* (3.26) *which is non-continuable to the left, where $\theta \in (0, 1)$. Then there exists $\varepsilon \in (j(\theta), 1)$ such that $r'(\varepsilon) = 0$.*

Proof of Lemma 3.7. By Proposition 1.41, $r'(R) \neq r(R)/R$ for all $R \in (j(\theta), 1]$.

Since $r'(1) = \theta < 1 = r(1)/1$, it follows that $r'(R) < r(R)/R$ for all $R \in (j(\theta), 1]$.

Now (3.21) shows that $r'' > 0$ on $(j(\theta), 1]$. Hence r' is a strictly increasing and continuous function on the interval $(j(\theta), 1]$. Note also that, since

$$r(R) - r(1) \geq (R-1)r'(1),$$

it follows that

$$r(R) > 1 - \theta \quad \text{for all } R \in (j(\theta), 1]. \tag{3.28}$$

Suppose, for a contradiction, that $r' \neq 0$ on $(j(\theta), 1]$. Since $r'(1) > 0$, it follows that $r' > 0$ on this interval. Then exactly one of the following two cases must occur: either $j(\theta) > 0$, or $j(\theta) = 0$.

Suppose first that $j(\theta) > 0$. Since both r and r' are increasing on the interval $(j(\theta), 1]$, there exist

$$\lim_{R \searrow j(\theta)} r'(R) =: l \geq 0,$$

and

$$\lim_{R \searrow j(\theta)} r(R) =: m \geq 1 - \theta > 0.$$

Since $(m, l) \in \mathbb{R}^2 \setminus \{0\}$, we have thus obtained a contradiction to the fact that r is non-continuable to the left.

Suppose now that $j(\theta) = 0$. Note that the conservation law (3.27) can be rewritten as

$$\begin{aligned} &R^{3-p}[(r'(R))^2R^2 + 2(r(R))^2]^{\frac{p}{2}-1}[(1-p)(r'(R))^2R^2 + 2(r(R))^2 + (p-3)r'r(R)R] \\ &\quad = (\theta^2+2)^{\frac{p}{2}-1}[(1-p)\theta^2 + 2 + (p-3)\theta]. \end{aligned} \tag{3.29}$$

Taking limits as $R \to 0$ in (3.29), we get that the left hand side tends to 0 while the right hand side is a non-zero constant, a contradiction which finishes the proof of the Lemma 3.7. $\square$

Proof of Theorem 3.5. Fix $\tilde{\beta} \in (0, 1)$. We consider solutions of (3.21) and (3.26) as θ varies in the interval $(0, 1)$. For each $\theta \in (0, 1)$, let r_θ be the solution of (3.21), (3.26), and let ε_θ be given by Lemma 3.7 such that $r'_\theta(\varepsilon_\theta) = 0$. Let $\beta_\theta := r_\theta(\varepsilon_\theta)$. The proof is finished once we show that there exists $\tilde{\theta} \in (0, 1)$ such that $\beta_{\tilde{\theta}} = \tilde{\beta}$, for then $\varepsilon_{\tilde{\theta}}$ and $r_{\tilde{\theta}}$ provide exactly what we were looking for. We do this by finding explicit formulae for β_θ and ε_θ in terms of θ.

Let $\theta \in (0,1)$ be fixed. For simplicity of notation, let $r := r_\theta$, $\varepsilon := \varepsilon_\theta$ and $\beta := \beta_\theta$. We study further properties of solutions of (3.21).

As noted in Subsection 1.3.6, the change of variables:

$$t = \frac{r(R)}{R}, \quad q(t) = r'(R), \tag{3.30}$$

leads to the ordinary differential equation (1.72) satisfied by q which, for Φ given by (3.19), takes the form

$$q'(t) = -2\frac{q^2 + 2t^2 + (p-2)qt}{q^2 + 2t^2 + (p-2)q^2}. \tag{3.31}$$

The solution q also satisfies the initial condition

$$q(1) = \theta.$$

Equation (3.31) is a 'homogeneous' equation, and the standard way to solve it is to consider $y(t) = q(t)/t$, which satisfies

$$\begin{aligned} y'(t) &= \frac{1}{t}\left(q'(t) - \frac{q(t)}{t}\right) = \frac{1}{t}\left(-2\frac{y^2 + (p-2)y + 2}{(p-1)y^2 + 2} - y\right) \\ &= -\frac{1}{t}\frac{(y^2+2)[(p-1)y+2]}{(p-1)y^2+2}. \end{aligned} \tag{3.32}$$

This is a 'separable' differential equation, which can be written in the form

$$\frac{dt}{t} = -G_p'(y)dy, \tag{3.33}$$

where G_p is such that

$$G_p'(y) = \frac{(p-1)y^2+2}{(y^2+2)[(p-1)y+2]}, \qquad \text{with } G_p(0) = 0. \tag{3.34}$$

Therefore, integrating (3.33), we obtain a first integral for the equation (3.31):

$$\log t + G_p\left(\frac{q(t)}{t}\right) = \text{constant}. \tag{3.35}$$

It follows that

$$\log\left(\frac{r(R)}{R}\right) + G_p\left(\frac{Rr'(R)}{r(R)}\right) = \text{constant}, \tag{3.36}$$

and hence, since $r(\varepsilon) = \beta$, $r'(1) = \theta$,

$$\log\beta - \log\varepsilon = G_p(\theta). \tag{3.37}$$

Also, using in (3.27) the fact that $r(\varepsilon) = \beta$, it follows that

$$\varepsilon^3\left(\frac{\beta}{\varepsilon}\right)^p = H_p(\theta), \tag{3.38}$$

where $H_p : (0,1) \to \mathbb{R}$ is given, for all $\theta \in (0,1)$, by

$$H_p(\theta) = 2^{-\frac{p}{2}}\left(\theta^2+2\right)^{\frac{p}{2}-1}(1-\theta)[(p-1)\theta+2]. \tag{3.39}$$

From (3.38) we obtain that

$$p\log\beta + (3-p)\log\varepsilon = \log H_p(\theta). \tag{3.40}$$

Using (3.37) and (3.40) we deduce the following explicit expressions for ε and β in terms of θ:

$$3\log\varepsilon = \log H_p(\theta) - pG_p(\theta), \tag{3.41}$$

$$3\log\beta = (3-p)G_p(\theta) + \log H_p(\theta). \tag{3.42}$$

Suppose now that θ varies in $(0,1)$. Consider the function $L_p : (0,1) \to \mathbb{R}$ given, for all $\theta \in (0,1)$, by

$$L_p(\theta) := (3-p)G_p(\theta) + \log H_p(\theta)$$

Note that, for any $\theta \in (0,1)$,

$$L_p'(\theta) = \frac{(p-2)\theta}{\theta^2+2} + \frac{p-1}{(p-1)\theta+2} - \frac{1}{1-\theta} + (3-p)\frac{(p-1)\theta^2+2}{(\theta^2+2)[(p-1)\theta+2]} \tag{3.43}$$
$$= \frac{-3\theta[(p-1)\theta^2+2]}{(1-\theta)(\theta^2+2)[(p-1)\theta+2]} < 0,$$

so that L_p is a strictly decreasing function on $(0,1)$. It is also easy to see that

$$\lim_{\theta\to 0} L_p(\theta) = 0 \quad \text{and} \quad \lim_{\theta\to 1} L_p(\theta) = -\infty.$$

It follows that L_p is a bijection from $(0,1)$ onto $(-\infty,0)$. We deduce from this and (3.42) that, for $\tilde{\beta} \in (0,1)$ considered at the beginning of this proof, there exists a (unique) $\tilde{\theta} \in (0,1)$ such that $\tilde{\beta} = \beta_{\tilde{\theta}}$. This completes the proof of Theorem 3.5.

□

Fix $\beta \in (0,1)$, let θ be the unique number in $(0,1)$ which satisfies (3.42), and consider $\tilde{r}_\theta$ the solution of (3.21) and (3.26). By Lemma 3.7, there exists $\varepsilon \in (0,1)$ such that $\tilde{r}_\theta'(\varepsilon) = 0$, and moreover $\tilde{r}_\theta(\varepsilon) = \beta$. Let $r_\beta : [0,1] \to [0,1]$ be given by

$$r_\beta(R) = \begin{cases} \tilde{r}_\theta(R), & \text{for } R \in [\varepsilon, 1], \\ \beta, & \text{for } R \in [0, \varepsilon]. \end{cases} \tag{3.44}$$

We denote by $\mathbf{u}_\beta$ the corresponding radial deformation

$$\mathbf{u}_\beta(\mathbf{x}) = r_\beta(|\mathbf{x}|)\frac{\mathbf{x}}{|\mathbf{x}|} \quad \text{for all } \mathbf{x} \in B(\mathbf{0},1). \tag{3.45}$$

3.3 The infimising property of $\mathbf{u}_\beta$

We now show that the function $\mathbf{u}_\beta$ whose existence was proved in the previous section infimises (in the sense of Theorem 3.8 below) the p-energy over the class $\mathcal{C}_V$. Note that $\mathbf{u}_\beta$ does not belong to $\mathcal{C}_V$, since $\det \nabla \mathbf{u}_\beta = 0$ everywhere in B_ε.

Theorem 3.8. *For every $\beta \in (0,1]$, let $V = 4\pi\beta^3/3$. Then the radial function*

$\mathbf{u}_\beta$ *given by* (3.45) *satisfies*

$$\inf\left\{\int_\Omega |\nabla \mathbf{u}|^p \, d\mathbf{x} : \; \mathbf{u} \in \mathcal{C}_V\right\} = \int_\Omega |\nabla \mathbf{u}_\beta|^p \, d\mathbf{x}, \tag{3.46}$$

where $\mathcal{C}_V$ *is the class of deformations defined in* (3.9).

Proof of Theorem 3.8. The first part of the proof consists in showing that, for every mapping $\mathbf{u} \in \mathcal{C}_V$,

$$\int_\Omega |\nabla \mathbf{u}_\beta|^p \, d\mathbf{x} \leq \int_\Omega |\nabla \mathbf{u}|^p \, d\mathbf{x}. \tag{3.47}$$

In the second part, we will exhibit a sequence $\{\hat{\mathbf{u}}_\delta\}_{\delta>0}$ of mappings in $\mathcal{C}_V$ such that

$$\lim_{\delta\searrow 0} \int_\Omega |\nabla \hat{\mathbf{u}}_\delta|^p \, d\mathbf{x} = \int_\Omega |\nabla \mathbf{u}_\beta|^p \, d\mathbf{x}. \tag{3.48}$$

The mappings $\{\hat{\mathbf{u}}_\delta\}_{\delta>0}$ will be chosen to be radial, with $\hat{\mathbf{u}}_\delta(\mathbf{x}) = \hat{r}_\delta(|\mathbf{x}|)\frac{\mathbf{x}}{|\mathbf{x}|}$, where $\{\hat{r}_\delta\}_{\delta>0}$ satisfy, for some $c_\delta > 0$,

$$\hat{r}_\delta(0) = \beta, \tag{3.49a}$$

$$0 < c_\delta \leq \frac{\hat{r}'_\delta(R)\hat{r}^2_\delta(R)}{R^2} \quad \text{on} \quad (0,1), \tag{3.49b}$$

$$\lim_{\delta\searrow 0} I(\hat{r}_\delta) = I(r_\beta), \tag{3.49c}$$

where I is given by (3.20).

Let $\mathbf{u} \in \mathcal{C}_V$, so that $\mathbf{u}$ opens a hole of volume $V = 4\pi\beta^3/3$ at the origin. Let ε be such that $r_\beta(R) = \beta$ on $[0,\varepsilon]$, and r_β satisfies the radial equilibrium equation on $[\varepsilon, 1]$. In order to prove (3.47), we show that

$$\int_{B_\varepsilon} |\nabla \mathbf{u}_\beta|^p \, d\mathbf{x} \leq \int_{B_\varepsilon} |\nabla \mathbf{u}|^p \, d\mathbf{x}, \tag{3.50}$$

and

$$\int_{\Omega\setminus B_\varepsilon} |\nabla \mathbf{u}_\beta|^p \, d\mathbf{x} \leq \int_{\Omega\setminus B_\varepsilon} |\nabla \mathbf{u}|^p \, d\mathbf{x}. \tag{3.51}$$

Here and in what follows, B_R denotes the ball $B(\mathbf{0}, R)$, for $R \in (0,1]$.

We now prove that (3.50) holds. Since $\mathbf{u} \in \mathcal{A}_{\mathrm{I},p}$, Proposition 1.38 shows that the distributional Jacobian $\mathrm{Det}\nabla\mathbf{u}$ is a Radon measure and

$$(\mathrm{Det}\nabla\mathbf{u})(B_R) = \mathcal{L}^3(\mathrm{im}_T(\mathbf{u}, B_R)) \quad \text{a.e. } R \in (0,1].$$

Using the above and the fact that $\mathbf{u} \in \mathcal{C}_V$, we obtain

$$V \leq \mathcal{L}^3(\mathrm{im}_T(\mathbf{u}, B_R)) \quad \text{a.e. } R \in (0,\varepsilon). \tag{3.52a}$$

On the other hand, since $\mathbf{u}_\beta$ is a radial mapping and $r_\beta(R) = \beta$ for all $R \in (0,\varepsilon)$, it follows that $\mathrm{im}_T(\mathbf{u}_\beta, B_R) = B_\beta$ for all $R \in (0,\varepsilon)$, and hence

$$V = \mathcal{L}^3(\mathrm{im}_T(\mathbf{u}_\beta, B_R)) \quad \text{for all } R \in [0,\varepsilon]. \tag{3.52b}$$

By using Propositions 1.25, 1.36 and 1.28 as in the proof of Theorem 2.1, it follows, see (2.11), that

$$\mathcal{L}^3(\mathrm{im}_T(\mathbf{u}, B_R))^{2/3} \leq c\int_{\partial B_R} |\nabla\mathbf{u}|^2\, d\mathcal{H}^2, \tag{3.53a}$$

where the constant c is given by

$$c = \frac{1}{6(\mathcal{L}^3(B_1))^{1/3}}.$$

It is also easy to check that

$$\mathcal{L}^3(\mathrm{im}_T(\mathbf{u}_\beta, B_R))^{2/3} = c\int_{\partial B_R} |\nabla\mathbf{u}_\beta|^2\, d\mathcal{H}^2. \tag{3.53b}$$

Now, by Hölder's Inequality it follows that, for a.e. $R \in (0,\varepsilon)$,

$$\int_{\partial B_R} |\nabla\mathbf{u}|^2\, d\mathcal{H}^2 \leq (\mathcal{H}^2(\partial B_R))^{1-\frac{2}{p}} \left(\int_{\partial B_R} |\nabla\mathbf{u}|^p\, d\mathcal{H}^2\right)^{\frac{2}{p}}. \tag{3.54a}$$

At the same time, since $|\nabla\mathbf{u}_\beta|^2$ is constant on ∂B_R for any $R \in (0,\varepsilon)$, it follows that, for any such R,

$$\int_{\partial B_R} |\nabla\mathbf{u}_\beta|^2\, d\mathcal{H}^2 = (\mathcal{H}^2(\partial B_R))^{1-\frac{2}{p}} \left(\int_{\partial B_R} |\nabla\mathbf{u}_\beta|^p\, d\mathcal{H}^2\right)^{\frac{2}{p}}. \tag{3.54b}$$

Combining (3.52)-(3.54) we conclude that, for almost every $R \in (0, \varepsilon)$,

$$\int_{\partial B_R} |\nabla \mathbf{u}_\beta|^p \, d\mathcal{H}^2 \leq \int_{\partial B_R} |\nabla \mathbf{u}|^p \, d\mathcal{H}^2. \tag{3.55}$$

Integrating with respect to R in (3.55) yields (3.50).

We now prove that (3.51) holds. For ease of notation we denote $W(\mathbf{F}) := |\mathbf{F}|^p$ for all $\mathbf{F} \in M^{3\times3}$. The convexity of W implies that

$$W(\nabla \mathbf{u}) \geq W(\nabla \mathbf{u}_\beta) + \frac{\partial W}{\partial \mathbf{F}}(\nabla \mathbf{u}_\beta) : [\nabla \mathbf{u} - \nabla \mathbf{u}_\beta]. \tag{3.56}$$

Integrating (3.56) on $\Omega \setminus B_\varepsilon$, the properties of $\mathbf{u}_\beta$ and the fact that $\mathbf{u}(\mathbf{x}) = \mathbf{u}_\beta(\mathbf{x})$ for all $\mathbf{x} \in \partial B_1$ imply, upon using Proposition 1.40, that (3.51) holds.

Since both (3.50) and (3.51) hold, it follows that (3.47) is satisfied. This finishes the first part of the proof of the theorem.

It remains to construct a sequence $\{\hat{r}_\delta\}_{\delta>0}$ satisfying (3.49). To this aim, let $\hat{r}_\delta : (0,1) \to \mathbb{R}$ be given, for $0 < \delta < 1 - \varepsilon$, by

$$\hat{r}_\delta(R) = \begin{cases} \left[\beta^3 + \left(\dfrac{r_\beta^3(\varepsilon+\delta) - \beta^3}{(\varepsilon+\delta)^3}\right) R^3\right]^{1/3}, & \text{for } R \in [0, \varepsilon+\delta), \\ r_\beta(R), & \text{for } R \in [\varepsilon+\delta, 1]. \end{cases} \tag{3.57}$$

Then clearly (3.49a) holds. Note also that

$$\frac{\hat{r}_\delta'(R)\hat{r}_\delta^2(R)}{R^2} = \frac{r_\beta^3(\varepsilon+\delta) - \beta^3}{(\varepsilon+\delta)^3} \quad \text{on} \quad (0, \varepsilon+\delta), \tag{3.58}$$

and

$$\frac{\hat{r}_\delta'(R)\hat{r}_\delta^2(R)}{R^2} = \frac{r_\beta'(R) r_\beta^2(R)}{R^2} \quad \text{on} \quad [\varepsilon+\delta, 1], \tag{3.59}$$

and, since r_β' is bounded away from 0 on $[\varepsilon+\delta, 1]$, we deduce (3.49b). The proof of (3.49c) is a straightforward application of the Dominated Convergence Theorem.

This completes the proof of Theorem 3.8. □

3.4 The value of Υ_p

It follows from (3.15) and the preceding results that

$$\Upsilon_p = \lim_{\beta \searrow 0} \frac{I(r_\beta) - I(id)}{\beta^3/3}, \tag{3.60}$$

where $id(R) = R$ for all $R \in [0,1]$. This section is devoted to calculating this limit.

To calculate the value of $I(r_\beta)$, we split the integral defining it into two parts,

$$\begin{aligned} I(r_\beta) = &\int_0^\varepsilon R^2 \Phi \left(r'_\beta(R), \frac{r_\beta(R)}{R}, \frac{r_\beta(R)}{R} \right) dR \\ &+ \int_\varepsilon^1 R^2 \Phi \left(r'_\beta(R), \frac{r_\beta(R)}{R}, \frac{r_\beta(R)}{R} \right) dR, \end{aligned} \tag{3.61}$$

where Φ is given by (3.19). To calculate the second integral in (3.61) we make use of the following identity, mentioned in Proposition 1.44, which is satisfied by solutions of the Euler-Lagrange equation (3.18):

$$\begin{aligned} \frac{d}{dR} &\left\{ R^3 \left[\Phi \left(r'(R), \frac{r(R)}{R}, \frac{r(R)}{R} \right) + \left(\frac{r(R)}{R} - r'(R) \right) \Phi_1 \left(r'(R), \frac{r(R)}{R}, \frac{r(R)}{R} \right) \right] \right\} \\ &= 3R^2 \Phi \left(r'(R), \frac{r(R)}{R}, \frac{r(R)}{R} \right). \end{aligned} \tag{3.62}$$

Integrating (3.62) from ε to 1 we obtain that

$$\begin{aligned} &\int_\varepsilon^1 R^2 \Phi \left(r'_\beta(R), \frac{r_\beta(R)}{R}, \frac{r_\beta(R)}{R} \right) dR \\ &= \frac{1}{3} \left[\Phi(\theta,1,1) + (1-\theta)\Phi_1(\theta,1,1) \right] - \frac{\varepsilon^3}{3} \Phi \left(0, \frac{\beta}{\varepsilon}, \frac{\beta}{\varepsilon} \right), \end{aligned} \tag{3.63}$$

since $\Phi_1(0, v_2, v_3) = 0$ for all v_2, v_3. On the other hand, since r_β is constant on $[0,\varepsilon]$, we obtain, using the fact that Φ is homogeneous of degree p, that

$$\begin{aligned} \int_0^\varepsilon R^2 \Phi \left(r'_\beta(R), \frac{r_\beta(R)}{R}, \frac{r_\beta(R)}{R} \right) dR &= \int_0^\varepsilon R^2 \Phi \left(0, \frac{\beta}{R}, \frac{\beta}{R} \right) dR \\ = \int_0^\varepsilon \varepsilon^p R^{2-p} \Phi \left(0, \frac{\beta}{\varepsilon}, \frac{\beta}{\varepsilon} \right) dR &= \frac{\varepsilon^3}{3-p} \Phi \left(0, \frac{\beta}{\varepsilon}, \frac{\beta}{\varepsilon} \right). \end{aligned} \tag{3.64}$$

Combining (3.63) and (3.64) yields:

$$I(r_\beta) = \frac{1}{3}\left[\Phi(\theta,1,1) + (1-\theta)\Phi_1(\theta,1,1)\right] + \frac{p}{3(3-p)}\Phi\left(0,\frac{\beta}{\varepsilon},\frac{\beta}{\varepsilon}\right). \tag{3.65}$$

Therefore, for every $\beta \in (0,1)$,

$$\begin{aligned}
&\frac{I(r_\beta) - I(id)}{\beta^3/3} \\
&= \frac{\Phi(\theta,1,1) - \Phi(1,1,1) + (1-\theta)\Phi_1(\theta,1,1)}{\beta^3} + \frac{p}{3-p}\frac{\varepsilon^3}{\beta^3}\Phi\left(0,\frac{\beta}{\varepsilon},\frac{\beta}{\varepsilon}\right) \\
&= \frac{\Phi(\theta,1,1) - \Phi(1,1,1) + (1-\theta)\Phi_1(\theta,1,1)}{\beta^3} + \frac{p}{3-p}2^{\frac{p}{2}}\left(\frac{\beta}{\varepsilon}\right)^{p-3}.
\end{aligned} \tag{3.66}$$

It is immediate from (3.42) and (3.37) that

$$\beta^3 = e^{(3-p)G_p(\theta)}H_p(\theta) \quad \text{and} \quad \frac{\beta}{\varepsilon} = e^{G_p(\theta)}. \tag{3.67}$$

Using (3.67) in (3.66) leads to

$$\begin{aligned}
&\frac{I(r_\beta) - I(id)}{\beta^3/3} \\
&= \frac{\Phi(\theta,1,1) - \Phi(1,1,1) + (1-\theta)\Phi_1(\theta,1,1)}{e^{(3-p)G_p(\theta)}H_p(\theta)} + \frac{p}{3-p}2^{\frac{p}{2}}\,e^{(p-3)G_p(\theta)} \\
&= \frac{\Phi(\theta,1,1) - \Phi(1,1,1) + (1-\theta)\Phi_1(\theta,1,1)}{1-\theta}\,\frac{2^{\frac{p}{2}}\,e^{(p-3)G_p(\theta)}}{(\theta^2+2)^{\frac{p}{2}-1}\left[(p-1)\theta+2\right]} \\
&\quad + \frac{p}{3-p}2^{\frac{p}{2}}\,e^{(p-3)G_p(\theta)},
\end{aligned} \tag{3.68}$$

where θ and β are related by (3.42). In order to find the value for Υ_p, one has to calculate the limit of the above expression as $\beta \searrow 0$. It is clear from (3.42) that, as $\beta \searrow 0$, one has that $\theta \nearrow 1$. Since

$$\lim_{\theta \nearrow 1}\left(\frac{\Phi(\theta,1,1) - \Phi(1,1,1)}{1-\theta} + \Phi_1(\theta,1,1)\right) = 0, \tag{3.69}$$

the relation (3.68) leads to the following result.

Theorem 3.9. *Let Υ_p by given by* (3.15). *Then*

$$\Upsilon_p = 2^{\frac{p}{2}} \frac{p}{3-p}\, e^{(p-3)G_p(1)}, \tag{3.70}$$

where the function G_p is given by (3.34).

As we now show, the value of $G_p(1)$, and hence that of Υ_p, can be determined explicitly.

Lemma 3.10. *Let the function G_p be given by* (3.34). *Then*

$$\begin{aligned} G_p(1) =& \frac{p+1}{(p-1)^2+2} \log(p+1) + \frac{(p-1)(p-2)}{2(p-1)^2+4} \log 3 \\ &- \frac{(p-1)^2+p+3}{2(p-1)^2+4} \log 2 + \frac{-2(p-2)}{(p-1)^2+2} \frac{1}{\sqrt{2}} \arctan \frac{1}{\sqrt{2}}. \end{aligned} \tag{3.71}$$

Proof of Lemma 3.10. By its definition,

$$G_p(1) = \int_0^1 \frac{(p-1)y^2+2}{(y^2+2)[(p-1)y+2]}\, dy. \tag{3.72}$$

Since

$$\begin{aligned} \frac{(p-1)y^2+2}{(y^2+2)[(p-1)y+2]} &= \frac{(p-1)(p+1)}{(p-1)^2+2} \frac{1}{(p-1)y+2} \\ &+ \frac{(p-1)(p-2)}{(p-1)^2+2} \frac{y}{y^2+2} + \frac{-2(p-2)}{(p-1)^2+2} \frac{1}{y^2+2}, \end{aligned}$$

the integral in (3.72) can be easily calculated and, after some re-arrangement, we obtain (3.71).

□

Remark 3.11. It is also interesting to note that the following limit is finite,

$$\lim_{\theta \to 1} \frac{\beta_\theta}{\varepsilon_\theta} = (p+1)^{\frac{p+1}{(p-1)^2+2}} \frac{3^{\frac{(p-1)(p-2)}{2(p-1)^2+4}}}{2^{\frac{(p-1)^2+p+3}{2(p-1)^2+4}}} \exp\left(\frac{-\sqrt{2}(p-2)}{(p-1)^2+2} \arctan \frac{1}{\sqrt{2}} \right). \tag{3.73}$$

In particular, for $p = 2$ the above limit is $3/2$, a value which can also be obtained from the explicit calculations in [41].

3.5 Optimality of Theorem 3.4

We now prove that the result of Theorem 3.4 is in a certain sense optimal.

Theorem 3.12. *Let $\gamma > \Upsilon_p$. Then for every $\lambda > 0$ there exists a function h with $\lambda^{3-p}h'(\lambda^3) = \gamma$ such that W_h is not $W^{1,p}$-quasiconvex at $\lambda\mathbf{I}$ over the class $\mathcal{A}^*_{\lambda\mathbf{I},p}$.*

Proof. Let $\gamma > \Upsilon_p$. From the formula

$$\Upsilon_p = \lim_{\beta\searrow 0} \frac{I(r_\beta) - I(id)}{\beta^3/3}, \tag{3.74}$$

we deduce that, for all β sufficiently small,

$$\frac{I(r_\beta) - I(id)}{\beta^3/3} < \gamma. \tag{3.75}$$

Fix a value of β such that (3.75) holds. Let $\{\hat{r}_\delta\}_{\delta>0}$ be such that (3.49) holds. It follows from (3.49c) that for all δ sufficiently small,

$$\frac{I(\hat{r}_\delta) - I(id)}{\beta^3/3} < \gamma. \tag{3.76}$$

Fix δ such that (3.76) holds. Let $C_\delta > 1$ be such that

$$\frac{\hat{r}'_\delta(R)\hat{r}^2_\delta(R)}{R^2} \leq C_\delta \quad \text{on } (0,1), \tag{3.77}$$

and let $\hat{\mathbf{u}}$ be the radial mapping associated to $\hat{r}_\delta$. Then

$$\frac{\int_\Omega |\nabla\hat{\mathbf{u}}|^p - |\nabla\mathbf{u}^{\text{hom}}_{\mathbf{I}}|^p \, d\mathbf{x}}{\int_\Omega \det\nabla\mathbf{u}^{\text{hom}}_{\mathbf{I}} - \det\nabla\hat{\mathbf{u}} \, d\mathbf{x}} < \gamma \tag{3.78}$$

and

$$0 < c_\delta \leq \det\nabla\hat{\mathbf{u}} \leq C_\delta. \tag{3.79}$$

Let λ be arbitrary and let $\alpha := \gamma/\lambda^{3-p}$. Let h satisfying (1.30) be such that $h(s) = \alpha s$ for all $s \in [c_\delta\lambda^3, C_\delta\lambda^3]$. With $\overline{\mathbf{u}} := \lambda\hat{\mathbf{u}}$, it can easily be checked that $E(\overline{\mathbf{u}}) < E(\mathbf{u}^{\text{hom}}_{\lambda\mathbf{I}})$, as required. □

3.6 The case of a finite number of holes

We now wish to extend the results of Theorem 3.1 and Theorem 3.4 to the case when the class of admissible deformations consists of those producing a finite number of holes in the material. We return to the proof of Theorem 3.1 with a view to extending the arguments therein.

Note that a consequence of (3.15) is that, for every $V \in (0, 4\pi/3]$,

$$V \leq \frac{1}{\Upsilon_p} \int_\Omega |\nabla \mathbf{u}_\beta|^p - |\nabla \mathbf{u}_\mathbf{I}^{\text{hom}}|^p \, d\mathbf{x}, \tag{3.80}$$

where $\beta \in (0, 1]$ is such that $V = 4\pi\beta^3/3$. Therefore, using (3.14), we get that for every $\mathbf{u} \in \mathcal{A}_{\mathbf{I},p}(\mathbf{0})$ with $m_\mathbf{u} = V\delta_\mathbf{0}$,

$$V \leq \frac{1}{\Upsilon_p} \int_\Omega |\nabla \mathbf{u}|^p - |\nabla \mathbf{u}_\mathbf{I}^{\text{hom}}|^p \, d\mathbf{x}. \tag{3.81}$$

We shall prove an analogue of (3.81) for mappings in the more general class of mappings producing a finite number of holes.

The crucial step in proving (3.14) was to show that

$$\int_\Omega |\nabla \mathbf{u}_\beta|^p \, d\mathbf{x} \leq \int_\Omega |\nabla \mathbf{u}|^p \, d\mathbf{x}, \tag{3.82}$$

for every $\mathbf{u} \in \mathcal{A}_{\mathbf{I},p}(\mathbf{0})$ with $m_\mathbf{u} = V\delta_\mathbf{0}$. For this, we made use of the Propositions 1.25, 1.36, 1.28 and 1.38 and Hölder's Inequality to obtain

$$\begin{aligned} V^{2/3} &\leq \mathcal{L}^3(\operatorname{im}_T(\mathbf{u}, B_R))^{2/3} \leq \omega \mathcal{H}^2(\mathbf{u}(\partial B_R)) \\ &\leq \frac{\omega}{2} \int_{\partial B_R} |\nabla \mathbf{u}|^2 \, d\mathcal{H}^2 \leq \frac{\omega}{2} (\mathcal{H}^2(\partial B_R))^{1-\frac{2}{p}} \left(\int_{\partial B_R} |\nabla \mathbf{u}|^p \, d\mathcal{H}^2 \right)^{\frac{2}{p}}, \end{aligned}$$

where B_R denotes the ball of radius R centered at $\mathbf{0}$, and $\omega = \mathcal{L}^3(B_1)^{-1/3}/3$, to deduce that

$$\int_{B_\varepsilon} |\nabla \mathbf{u}_\beta|^p \, d\mathbf{x} \leq \int_{B_\varepsilon} |\nabla \mathbf{u}|^p \, d\mathbf{x}.$$

On the other hand,

$$\int_{\Omega \setminus B_\varepsilon} |\nabla \mathbf{u}_\beta|^p \, d\mathbf{x} \leq \int_{\Omega \setminus B_\varepsilon} |\nabla \mathbf{u}|^p \, d\mathbf{x}.$$

since $W_0(\mathbf{F}) := |\mathbf{F}|^p$ is convex, $\mathbf{u}_\beta$ satisfies the Euler-Lagrange equations for W_0 on $\Omega \setminus B_\varepsilon$ and natural boundary conditions on ∂B_ε.

It is clear that the above argument extends to show that (3.82) is valid for all $\mathbf{u}$ such that

$$m_{\mathbf{u}} = V\delta_0 + \widetilde{m}_{\mathbf{u}}, \tag{3.83}$$

where $\widetilde{m}_{\mathbf{u}}$ is a nonnegative measure. Together with (3.80) this shows that (3.81) holds for all mappings $\mathbf{u}$ satisfying (3.83). Moreover, a standard scaling argument shows that (3.81) holds for all mappings $\mathbf{u}$ with

$$m_{\mathbf{u}} = V\delta_{\mathbf{a}} + \widetilde{m}_{\mathbf{u}}, \tag{3.84}$$

where $V \in (0, 4\pi/3]$, $\mathbf{a} \in \Omega$ and $\widetilde{m}_{\mathbf{u}}$ is a nonnegative measure.

Consider now mappings $\mathbf{u}$ such that the singular part of the distributional Jacobian of $\mathbf{u}$ is a finite combination of Dirac masses supported at N points in the domain, i.e.

$$m_{\mathbf{u}} = V_1\delta_{\mathbf{a}_1} + \ldots + V_N\delta_{\mathbf{a}_N} \tag{3.85}$$

where $\mathbf{a}_1, \ldots, \mathbf{a}_N \in \Omega$. We deduce from the above considerations that, for such mappings $\mathbf{u}$,

$$V_1 \leq \frac{1}{\Upsilon_p}\int_\Omega |\nabla\mathbf{u}|^p - |\nabla\mathbf{u}_{\mathbf{I}}^{\mathrm{hom}}|^p\,d\mathbf{x},$$

$$V_2 \leq \frac{1}{\Upsilon_p}\int_\Omega |\nabla\mathbf{u}|^p - |\nabla\mathbf{u}_{\mathbf{I}}^{\mathrm{hom}}|^p\,d\mathbf{x},$$

$$\ldots$$

$$V_N \leq \frac{1}{\Upsilon_p}\int_\Omega |\nabla\mathbf{u}|^p - |\nabla\mathbf{u}_{\mathbf{I}}^{\mathrm{hom}}|^p\,d\mathbf{x}.$$

By adding the previous relations we obtain that

$$m_{\mathbf{u}}(\overline{\Omega}) \leq \frac{N}{\Upsilon_p}\int_\Omega |\nabla\mathbf{u}|^p - |\nabla\mathbf{u}_{\mathbf{I}}^{\mathrm{hom}}|^p\,d\mathbf{x}. \tag{3.86}$$

This shows that

$$\alpha\lambda^{3-p} \leq \frac{\Upsilon_p}{N}$$

is a sufficient condition for the $W^{1,p}$-quasiconvexity at $\lambda\mathbf{I}$ over the class of functions $\mathbf{u} \in \mathcal{A}_{\lambda\mathbf{I},p}$ satisfying (3.85) of the model energy function W_α.

It also follows from this that

$$h'(\lambda^3)\lambda^{3-p} \leq \frac{\Upsilon_p}{N}$$

is a sufficient condition for the $W^{1,p}$-quasiconvexity at $\lambda\mathbf{I}$ over the class of functions $\mathbf{u} \in \mathcal{A}_{\lambda\mathbf{I},p}$ satisfying (3.85) of the stored energy function W_h.

Whilst the estimate (3.86) is sharp for $N = 1$, and might be considered reasonably good for N small, it is certainly not satisfactory for N very large. In fact, for all N large enough, (3.86) is worse than (2.7).

Chapter 4

Necessary Conditions for $W^{1,p}$-quasiconvexity

In this chapter we study necessary conditions for $W^{1,p}$-quasiconvexity of the stored energy functions given by

$$W_h(\mathbf{F}) = |\mathbf{F}|^p + h(\det \mathbf{F}) \qquad \text{for all } \mathbf{F} \in M_+^{3\times 3},$$

and

$$W_\alpha(\mathbf{F}) = |\mathbf{F}|^p + \alpha \det \mathbf{F} \qquad \text{for all } \mathbf{F} \in M_+^{3\times 3},$$

where $p \in [2,3)$ and h satisfies (1.30).

4.1 Necessary condition for $W^{1,p}$-quasiconvexity of W_h at a matrix $\mathbf{A}$

We start with a necessary condition for the $W^{1,p}$-quasiconvexity of W_h at a matrix $\mathbf{A}$ in the class $\mathcal{A}^*_{\mathbf{A},p}$.

Theorem 4.1. *For $2 \le p < 3$, let Λ_p be given by*

$$\Lambda_p := \frac{1}{\omega_3}\int_{\mathbb{R}^3}\left\{\left[\frac{y_1^2}{|\mathbf{y}|^2}\left(1+\frac{1}{|\mathbf{y}|^3}\right)^{-4/3} + \frac{y_2^2+y_3^2}{|\mathbf{y}|^2}\left(1+\frac{1}{|\mathbf{y}|^3}\right)^{2/3}\right]^{p/2} - 1\right\} d\mathbf{y} - \frac{p}{3}, \tag{4.1}$$

where $\omega_3 = 4\pi/3$ is the volume of the unit ball in $\mathbb{R}^3$. If the stored energy function W_h is $W^{1,p}$-quasiconvex at $\mathbf{A}$ in the class $\mathcal{A}^*_{\mathbf{A},p}$, then

$$(\det \mathbf{A})h'(\det \mathbf{A}) \leq \Lambda_p |\mathbf{A}|^p. \tag{4.2}$$

Remark 4.2. It is elementary to check that the integral in the definition of Λ_p converges.

Proof of Theorem 4.1. By Propositions 1.50 and 1.48, it suffices to consider the case of diagonal matrices $\mathbf{D}$ with entries λ_1, λ_2, λ_3, and when the domain Ω is the unit ball.

For any $\mathbf{u} \in \mathcal{A}_{\mathbf{D},p}(\mathbf{0})$, let us write $\mathbf{u} = \mathbf{D}\mathbf{v}$, where $\mathbf{v} \in \mathcal{A}_{\mathbf{I},p}(\mathbf{0})$. Let $\mathbf{v} = (v_1, v_2, v_3)$. Then

$$E(\mathbf{u}) = \int_\Omega (\lambda_1^2|\nabla v_1|^2 + \lambda_2^2|\nabla v_2|^2 + \lambda_3^2|\nabla v_3|^2)^{p/2} + h(\lambda_1\lambda_2\lambda_3 \det \nabla \mathbf{v})\, d\mathbf{x}. \tag{4.3}$$

Since the function $f : [0,\infty) \to [0,\infty)$ given by $f(t) = t^{p/2}$ is convex, it follows that, for every $t_1, t_2, t_3 \geq 0$,

$$(\lambda_1^2 t_1 + \lambda_2^2 t_2 + \lambda_3^2 t_3)^{p/2} \leq (\sum_{i=1}^3 \lambda_i^2)^{p/2} \left(\frac{\lambda_1^2}{\sum_{i=1}^3 \lambda_i^2} t_1^{p/2} + \frac{\lambda_2^2}{\sum_{i=1}^3 \lambda_i^2} t_2^{p/2} + \frac{\lambda_3^2}{\sum_{i=1}^3 \lambda_i^2} t_3^{p/2} \right). \tag{4.4}$$

Suppose now that $\mathbf{v} \in \mathcal{A}_{\mathbf{I},p}(\mathbf{0})$ is any radial mapping, and let $\mathbf{u} = \mathbf{D}\mathbf{v}$. It follows from (4.3) and (4.4) that, for any such $\mathbf{u}$,

$$\begin{aligned} E(\mathbf{u}) &\leq (\sum_{j=1}^3 \lambda_j^2)^{p/2} \left(\sum_{j=1}^3 \frac{\lambda_j^2}{\sum_{i=1}^3 \lambda_i^2} \int_\Omega |\nabla v_j|^p\, d\mathbf{x} \right) + \int_\Omega h(\lambda_1\lambda_2\lambda_3 \det \nabla \mathbf{v})\, d\mathbf{x}, \\ &= (\sum_{j=1}^3 \lambda_j^2)^{p/2} \int_\Omega |\nabla v_1|^p\, d\mathbf{x} + \int_\Omega h(\lambda_1\lambda_2\lambda_3 \det \nabla \mathbf{v})\, d\mathbf{x} \end{aligned} \tag{4.5}$$

where we have used the equality

$$\int_\Omega |\nabla v_1|^p\, d\mathbf{x} = \int_\Omega |\nabla v_2|^p\, d\mathbf{x} = \int_\Omega |\nabla v_3|^p\, d\mathbf{x},$$

which is an immediate consequence of the fact that $\mathbf{v}$ is radial. Since W_h is $W^{1,p}$

quasiconvex at $\mathbf{D}$, it follows that

$$0 \leq E(\mathbf{u}) - E(\mathbf{u}_{\mathbf{D}}^{\mathrm{hom}}) \leq |\mathbf{D}|^p \int_\Omega |\nabla v_1|^p - 1 \, d\mathbf{x} + \int_\Omega h(\det \mathbf{D} \det \nabla \mathbf{v}) - h(\det \mathbf{D}) \, d\mathbf{x}. \tag{4.6}$$

Note that, if $\mathbf{v}(\mathbf{x}) = r(R)\frac{\mathbf{x}}{R}$ is radial, then

$$\nabla \mathbf{v}(\mathbf{x}) = \frac{r(R)}{R}\mathbf{I} + \left(r'(R) - \frac{r(R)}{R}\right) \frac{\mathbf{x} \otimes \mathbf{x}}{R^2}, \tag{4.7}$$

so that, by an easy calculation,

$$|\nabla v_1|^2 = \frac{x_1^2}{R^2}(r'(R))^2 + \frac{x_2^2 + x_3^2}{R^2} \left(\frac{r(R)}{R}\right)^2 \tag{4.8}$$

For $a \in (0, 1]$, let $\mathbf{v}_a : B(\mathbf{0}, 1) \to \mathbb{R}^3$ be the radial deformations given by

$$\mathbf{v}_a(\mathbf{x}) = r_a(|\mathbf{x}|)\frac{\mathbf{x}}{|\mathbf{x}|},$$

where

$$r_a(R) = \left(aR^3 + (1-a)\right)^{1/3} \quad \text{for all } a \in (0, 1].$$

It follows from (4.6) that, for all $a \in (0, 1]$,

$$0 \leq |\mathbf{D}|^p \int_\Omega |\nabla v_{a,1}|^p - 1 \, d\mathbf{x} + \int_\Omega h(\det \mathbf{D} \det \nabla \mathbf{v}_a) - h(\det \mathbf{D}) \, d\mathbf{x}. \tag{4.9}$$

We shall prove that the following limit exists

$$m_p := \lim_{a \nearrow 1} \frac{|\mathbf{D}|^p \int_\Omega |\nabla v_{a,1}|^p - 1 \, d\mathbf{x} + \int_\Omega h(\det \mathbf{D} \det \nabla \mathbf{v}_a) - h(\det \mathbf{D}) \, d\mathbf{x}}{\omega_3(1-a)}, \tag{4.10}$$

and then it will follow from (4.9) that necessarily $m_p \geq 0$.

Using (4.8), it follows that

$$\begin{aligned}\int_{B_1} &|\nabla v_{a,1}|^p - 1\,d\mathbf{x}\\ &= a^{p/3}\int_{B_1}\left\{\left[\frac{x_1^2}{|\mathbf{x}|^2}\left(1+\frac{1-a}{a|\mathbf{x}|^3}\right)^{-4/3}+\frac{x_2^2+x_3^2}{|\mathbf{x}|^2}\left(1+\frac{1-a}{a|\mathbf{x}|^3}\right)^{2/3}\right]^{p/2}-1\right\}d\mathbf{x}\\ &\quad+(a^{p/3}-1)\omega_3 \qquad (4.11)\end{aligned}$$

Upon making the change of variables $\mathbf{y} := \left(\frac{a}{1-a}\right)^{1/3}\mathbf{x}$, and letting $a \nearrow 1$, we deduce that

$$m_p = \Lambda_p|\mathbf{D}|^p - (\det\mathbf{D})h'(\det\mathbf{D}),$$

where Λ_p is given by (4.1). Since $m_p \geq 0$, the required conclusion follows. □

4.2 Necessary condition for $W^{1,p}$-quasiconvexity of W_h at a matrix $\lambda\mathbf{I}$

We now show that, for matrices of the form $\lambda\mathbf{I}$, the result of Theorem 4.1 can be improved. Instead of the class $\mathcal{A}^*_{\lambda\mathbf{I},p}$, it suffices to restrict attention to the class of radial deformations. Although the same result can be obtained by the method of Stuart [50], the proof given here is more elementary.

Theorem 4.3. *For $1 < p < 3$, let Γ_p be given by*

$$\Gamma_p := \int_0^\infty\left\{\left[\frac{1}{3}\left(1+\frac{1}{s}\right)^{-4/3}+\frac{2}{3}\left(1+\frac{1}{s}\right)^{2/3}\right]^{p/2}-1\right\}ds-\frac{p}{3}. \qquad (4.12)$$

If the stored energy function W_h is $W^{1,p}$-quasiconvex at $\lambda\mathbf{I}$ in the class of radial deformations, then

$$\lambda^{3-p}h'(\lambda^3) \leq 3^{p/2}\Gamma_p. \qquad (4.13)$$

Remark 4.4. It is elementary to check that the integral in the definition of Γ_p converges.

Proof of Theorem 4.3. For $a \in (0,1]$, let $\mathbf{u}_a : B(\mathbf{0},1) \to \mathbb{R}^3$ be the radial deformations given by

$$\mathbf{u}_a(\mathbf{x}) = r_a(|\mathbf{x}|)\frac{\mathbf{x}}{|\mathbf{x}|},$$

where

$$r_a(R) = \lambda\big(aR^3 + (1-a)\big)^{1/3} \quad \text{for any } a \in (0,1].$$

Note that $\mathbf{u}_1 = \mathbf{u}_{\lambda\mathbf{I}}^{\text{hom}}$. Then $E(\mathbf{u}_a) = 4\pi I(r_a)$, where

$$I(r) = \int_0^1 R^2 \left[\left((r'(R))^2 + 2\left(\frac{r(R)}{R}\right)^2 \right)^{p/2} + h\left(\frac{r'(R)r^2(R)}{R^2}\right) \right] dR.$$

We shall prove that the following limit exists

$$l_p := \lim_{a \nearrow 1} \frac{I(r_a) - I(r_1)}{1-a}. \tag{4.14}$$

Since W_h is $W^{1,p}$-quasiconvex at $\lambda\mathbf{I}$, it follows in particular that $E(\mathbf{u}_a) \geq E(\mathbf{u}_1)$ for all $a \in (0,1)$. This implies that necessarily $l_p \geq 0$. We shall calculate l_p explicitly, and the inequality $l_p \geq 0$ will turn out to be equivalent to (4.13).

For any fixed $a \in (0,1)$, the following holds:

$$\begin{aligned} I(r_a) - I(r_1) &= \int_0^1 R^2 \left[\left((r_a'(R))^2 + 2\left(\frac{r_a(R)}{R}\right)^2 \right)^{p/2} - 3^{p/2}\lambda^p \right] dR \qquad (4.15) \\ &\quad + \frac{1}{3}\left[h(\lambda^3 a) - h(\lambda^3)\right] \\ &= 3^{p/2}\lambda^p J + \frac{1}{3}\left[h(\lambda^3 a) - h(\lambda^3)\right], \qquad (4.16) \end{aligned}$$

where

$$J := \int_0^1 R^2 \left[\left(\frac{1}{3}\frac{a^2R^4}{(aR^3 + (1-a))^{4/3}} + \frac{2}{3}\frac{(aR^3 + (1-a))^{2/3}}{R^2} \right)^{p/2} - 1 \right] dR.$$

Note now that

$$J = a^{p/3}\int_0^1 R^2\left[\left(\frac{1}{3}\left(1+\frac{1-a}{aR^3}\right)^{-4/3}+\frac{2}{3}\left(1+\frac{1-a}{aR^3}\right)^{2/3}\right)^{p/2}-1\right]dR \tag{4.17}$$

$$+\frac{1}{3}(a^{p/3}-1). \tag{4.18}$$

Upon making the change of variables

$$\frac{aR^3}{1-a}=s, \qquad \text{so that} \qquad 3R^2\,dR=\frac{1-a}{a}\,ds,$$

in the integral, it follows that

$$J=\frac{(1-a)a^{p/3}}{3a}\int_0^{a/(1-a)}\left[\left(\frac{1}{3}\left(1+\frac{1}{s}\right)^{-4/3}+\frac{2}{3}\left(1+\frac{1}{s}\right)^{2/3}\right)^{p/2}-1\right]ds$$
$$+\frac{1}{3}(a^{p/3}-1).$$

Using this equality in (4.16), and letting $a \nearrow 1$, we conclude that

$$l_p = 3^{p/2}\lambda^p\left\{\frac{1}{3}\int_0^\infty\left[\left(\frac{1}{3}\left(1+\frac{1}{s}\right)^{-4/3}+\frac{2}{3}\left(1+\frac{1}{s}\right)^{2/3}\right)^{p/2}-1\right]ds-\frac{p}{9}\right\}$$
$$-\frac{1}{3}\lambda^3 h'(\lambda^3). \tag{4.19}$$

Note that

$$l_p=\frac{1}{3}\lambda^p[3^{p/2}\Gamma_p-\lambda^{3-p}h'(\lambda^3)]. \tag{4.20}$$

Since the condition $l_p \geq 0$ is obviously equivalent to $\lambda^{3-p}h'(\lambda^3) \leq 3^{p/2}\Gamma_p$, this completes the proof of Theorem 4.3. □

We now calculate explicitly the value of Γ_2.

Proposition 4.5. *Let* Γ_2 *be given by* (4.12). *Then* $\Gamma_2 = 1$.

We do this by calculating explicitly the value of $I(r_a)$, for all $a \in (0,1)$. To this aim, we now calculate integrals of the type

$$K(r)=\int_0^A (r'(R))^2R^2+2r^2(R)\,dR$$

for functions $r : [0, \infty) \to [0, \infty)$ of the form

$$r(R) = (\alpha^3 + aR^3)^{1/3}, \tag{4.21}$$

where $\alpha > 0$ and $a \geq 0$. These results will be useful later on in this chapter.

Note that, for r given by (4.21),

$$r'(R)r^2(R) = aR^2 \quad \text{for all } R \in [0, \infty).$$

Using this and integration by parts, we obtain

$$\begin{aligned}
\int_0^A (r'(R))^2 R^2 \, dR &= \int_0^A \frac{aR^4 r'(R)}{r^2(R)} \, dR = -\int_0^A aR^4 \left(\frac{1}{r(R)}\right)' \, dR \\
&= -\frac{aA^4}{r(A)} + \int_0^A \frac{4aR^3}{r(R)} \, dR = -\frac{aA^4}{r(A)} + \int_0^A 2R[r^2(R)]' \, dR \\
&= -\frac{aA^4}{r(A)} + 2Ar^2(A) - \int_0^A 2r^2(R) \, dR.
\end{aligned}$$

We deduce that, for r given by (4.21),

$$K(r) = -\frac{aA^4}{r(A)} + 2Ar^2(A). \tag{4.22}$$

Proof of Proposition 4.5. It follows from (4.22) that

$$I(r_a) = \lambda^2(2 - a) + \frac{1}{3}h(\lambda^3 a) \quad \text{for all } a \in (0, 1).$$

We deduce from (4.14) that

$$l_2 = \lambda^2 - \frac{1}{3}\lambda^3 h'(\lambda^3). \tag{4.23}$$

Comparing this formula with (4.20), we conclude that $\Gamma_2 = 1$, as required. $\square$

4.3 Optimality of the necessary condition for $W^{1,2}$-quasiconvexity of W_h in the radial case

We now prove that, when $p = 2$, the results of Theorem 4.3 and Proposition 4.5 are optimal, in the sense that, under the given hypothesis, one cannot replace the value of $\Gamma_2 = 1$ in the conclusion of the Proposition 4.5 by any smaller constant.

Theorem 4.6. *For every $\lambda \in (0, \infty)$ and for every $\gamma \in (-\infty, 3)$, there exists a function h satisfying (1.30) such that $\lambda h'(\lambda^3) > \gamma$ and $E(\mathbf{u}_{\lambda\mathbf{I}}^{\mathrm{hom}}) \leq E(\mathbf{u})$ for all radial mappings $\mathbf{u}$ with $\mathbf{u}(\mathbf{x}) = \lambda\mathbf{x}$ on $\partial\Omega$.*

We start with some general considerations which will be useful for the proof of Theorem 4.6. These involve solving explicitly the problem of minimizing the functional

$$I(r) := \int_0^1 (r'(R))^2 R^2 + 2r^2(R)\, dR, \tag{4.24}$$

in the class

$$\mathcal{C}_{\alpha,a} := \left\{ r \in W^{1,1}(0,1) : r(0) = \alpha,\ r(1) = 1,\ r' r^2 \geq aR^2 \text{ a.e. } R \in (0,1) \right\}, \tag{4.25}$$

where α and a satisfy the compatibility condition

$$a + \alpha^3 \leq 1. \tag{4.26}$$

Formally, the Euler-Lagrange equation for the functional I is given by:

$$\frac{d}{dR}[R^2 r'] = 2r. \tag{4.27}$$

The solutions of (4.27) are of the form

$$r(R) = cR + \frac{d}{2R^2},$$

where c, $d \in \mathbb{R}$ are constants.

The following two lemmas generalize some results in [41], where $a = 0$.

Lemma 4.7. *For every α and a satisfying (4.26), there exist unique $A \in (0, 1]$*

and $c, d \in \mathbb{R}$ such that the function $\tilde{r}_{\alpha,a} : [0,1] \to \mathbb{R}$ given by

$$\tilde{r}_{\alpha,a}(R) = \begin{cases} (\alpha^3 + aR^3)^{1/3}, & \text{for all } R \in [0, A], \\ cR + \dfrac{d}{2R^2}, & \text{for all } R \in [A, 1], \end{cases} \tag{4.28}$$

is of class C^1 and belongs to $\mathcal{C}_{\alpha,a}$. Moreover, if we denote

$$p_{\alpha,a} := c + \frac{d}{2A^3}, \tag{4.29}$$

then $p_{\alpha,a} \in [1, 3/2]$ and satisfies

$$3p_{\alpha,a}^2 - 2p_{\alpha,a}^3 = \alpha^3 + a. \tag{4.30}$$

Remark 4.8. Note that the function $\tilde{r}_{\alpha,a}$ in Lemma 4.7 has the property that

$$\frac{\tilde{r}'_{\alpha,a}\tilde{r}^2_{\alpha,a}}{R^2} = a \quad \text{for all } R \in [0, A],$$

which means that the corresponding radial mapping $\tilde{\mathbf{u}}_{\alpha,a}(\mathbf{x}) = \lambda \tilde{r}_{\alpha,a}(|\mathbf{x}|)\frac{\mathbf{x}}{|\mathbf{x}|}$ has constant determinant $\lambda^3 a$ for $|\mathbf{x}| \in [0, A]$.

Lemma 4.9. *For every α and a satisfying* (4.26)*, the function $\tilde{r}_{\alpha,a}$ in Lemma 4.7 is a minimiser of I on $\mathcal{C}_{\alpha,a}$.*

Proof of Lemma 4.7. Fix α and a satisfying (4.26). For convenience of notation, during this proof $p_{\alpha,a}$ given by (4.29) will be denoted by p. The requirements that $\tilde{r}_{\alpha,a}$ is a C^1 function in $\mathcal{C}_{\alpha,a}$ are expressed by the following equations:

$$cA + \frac{d}{2A^2} = (\alpha^3 + aA^3)^{1/3}, \tag{4.31a}$$

$$\left(c - \frac{d}{A^3}\right)\left(c + \frac{d}{2A^3}\right)^2 = a, \tag{4.31b}$$

$$c + \frac{d}{2} = 1. \tag{4.31c}$$

We now prove that the system of equations (4.31) for the unknowns c, $d \in \mathbb{R}$ and $A \in (0, 1]$ has a unique solution.

We prove first that there exists at most one solution of (4.31). Let c, $d \in \mathbb{R}$

and $A \in (0, 1]$ satisfy (4.31). Let

$$p := c + \frac{d}{2A^3}, \tag{4.32}$$

which is in agreement with (4.29). It follows from (4.31a) that

$$A^3(p^3 - a) = \alpha^3. \tag{4.33}$$

Also, it follows from (4.31b) that

$$c - \frac{d}{A^3} = \frac{a}{p^2}. \tag{4.34}$$

We deduce from (4.32) and (4.34) that

$$3c = 2p + \frac{a}{p^2}, \tag{4.35}$$

$$\frac{3d}{2} = A^3\left(p - \frac{a}{p^2}\right), \tag{4.36}$$

and therefore, using (4.31c), that

$$3 = 2p + \frac{a}{p^2} + A^3\left(p - \frac{a}{p^2}\right). \tag{4.37}$$

It follows, upon using (4.33) in (4.37), that p satisfies

$$3p^2 - 2p^3 = a + \alpha^3. \tag{4.38}$$

Note now from (4.33) that, since $A \leq 1$, it follows that $p^3 \geq a + \alpha^3$, and we can deduce from (4.38) that $p^2 \leq p^3$, and hence $p \geq 1$.

Since (4.26) holds, the equation (4.38) has a unique solution in $[1, \infty)$ and, moreover, this solution belongs to $[1, 3/2]$. Therefore, the value of p is uniquely determined, namely as the unique solution of (4.38). The value of A now follows from (4.33), and then the values of c and d follow from (4.35) and (4.36). This completes the proof of uniqueness of solutions of (4.31).

We now sketch the proof of existence of solutions of (4.31). Let p be the unique solution in $[1, 3/2]$ of (4.38). One defines A to be such that (4.33) holds, and then c and d such that (4.35) and (4.36) are satisfied. It is not difficult, although a bit tedious, to check that A, c and d constructed in this way satisfy (4.31), but we omit the details.

Note that we have already checked during the proof that $p_{a,\alpha}$ given by (4.29) belongs to the interval $[1, 3/2]$ and satisfies (4.30). The proof of Lemma 4.7 is therefore complete.

$\square$

Proof of Lemma 4.9. For convenience of notation, during this proof the function $\tilde{r}_{\alpha,a}$ in (4.28) will be denoted by $\tilde{r}$. We shall prove that

$$I(\tilde{r}) \leq I(r) \quad \text{for all mappings } r \in \mathcal{C}_{\alpha,a}. \tag{4.39}$$

Let $r \in \mathcal{C}_{\alpha,a}$ be of the form $r = \tilde{r} + \varphi$. Then

$$I(r) = \int_0^1 \tilde{r}'^2 R^2 + 2\tilde{r}^2 + 2R^2\tilde{r}'\varphi' + 4\tilde{r}\varphi + R^2(\varphi')^2 + 2\varphi^2 \, dR.$$

Integrating by parts the third term in the above sum, we get that

$$I(r) = I(\tilde{r}) + 2\int_0^1 \left(-\frac{d}{dR}[R^2\tilde{r}'] + 2\tilde{r}\right)\varphi \, dR + \int_0^1 R^2(\varphi')^2 + 2\varphi^2 \, dR.$$

Hence, using the fact that $\tilde{r}$ satisfies (4.27) on $[A, 1]$, we obtain

$$I(r) = I(\tilde{r}) + 2\int_0^A \left(-\frac{d}{dR}[R^2\tilde{r}'] + 2\tilde{r}\right)\varphi \, dR + \int_0^1 R^2(\varphi')^2 + 2\varphi^2 \, dR.$$

Since the third term of the above sum is non-negative, the proof of (4.39) will be completed once we show that the second term is non-negative. To do this, we will show that on the interval $[0, A]$, both φ and

$$-\frac{d}{dR}[R^2\tilde{r}'] + 2\tilde{r}$$

are non-negative. Indeed, since $\varphi = r - \tilde{r}$, where r and $\tilde{r}$ satisfy

$$\tilde{r}'\tilde{r}^2 = aR^2 \ \text{ and } r'r^2 \geq aR^2 \text{ on } [0, A],$$

it follows that

$$r'r^2 - \tilde{r}'\tilde{r}^2 = \frac{1}{3}\frac{d}{dR}(r^3 - \tilde{r}^3) \geq 0,$$

so that $r^3 - \tilde{r}^3$ is increasing, with $r^3(0) - \tilde{r}^3(0) = 0$. It follows from this that $r^3(R) \geq \tilde{r}^3(R)$ for all $R \in [0, A]$, so that $\varphi = r - \tilde{r} \geq 0$ on $[0, A]$. Also, since

$$\tilde{r}(R) = (\alpha^3 + aR^3)^{1/3} \quad \text{for all } R \in [0, A],$$

it follows that

$$\begin{aligned} -\frac{d}{dR}[R^2\tilde{r}'] + 2\tilde{r} &= -2R\tilde{r}' - R^2\tilde{r}'' + 2\tilde{r} \\ &= 2(\alpha^3 + aR^3)^{1/3}\left(\frac{\alpha^3}{\alpha^3 + aR^3}\right)^2 \\ &\geq 0, \quad \text{on} \quad [0, A]. \end{aligned}$$

This completes the proof of the Lemma 4.9. □

Proof of Theorem 4.6. We assume, with no loss of generality, that $\gamma > 0$. Then, for any $\lambda \in (0, \infty)$ and $a \in (0, 1)$, there exists a convex function h satisfying (1.30) and such that

$$\gamma < \lambda h'(\lambda^3) < \frac{1}{2}(\gamma + 3), \quad h'(\lambda^3 a) = 0 \quad \text{and} \quad h'(s) \neq 0 \quad \text{for all } s \neq \lambda^3 a. \tag{4.40}$$

We shall prove that there exists $a \in (0, 1)$ such that, for any convex function h satisfying (1.30) and (4.40), the corresponding energy E satisfies $E(\mathbf{u}_{\lambda\mathbf{I}}^{\mathrm{hom}}) \leq E(\mathbf{u})$ for all radial maps $\mathbf{u}$.

For proving this claim, we argue by contradiction and assume that, for every $a \in (0, 1)$ there exists a convex function h satisfying (1.30) and (4.40) such that there exists a radial mapping $\bar{\mathbf{u}}$ with $E(\bar{\mathbf{u}}) < E(\mathbf{u}_{\lambda\mathbf{I}}^{\mathrm{hom}})$. By results of Ball [8], E has a minimiser $\tilde{\mathbf{u}}_0$ in the class of radial deformations, $\tilde{\mathbf{u}}_0(\mathbf{x}) = \tilde{r}_0(|\mathbf{x}|)\frac{\mathbf{x}}{|\mathbf{x}|}$, where necessarily $\tilde{r}_0(0) > 0$. Moreover, $\tilde{r}_0$ satisfies the radial Euler-Lagrange equation, and

$$\lim_{R \searrow 0} T(\tilde{r}_0(R)) = 0, \tag{4.41}$$

where

$$T(r(R)) = \left(\frac{R}{r(R)}\right)^2 \Phi_1\left(r'(R), \frac{r(R)}{R}, \frac{r(R)}{R}\right) \tag{4.42}$$

is the radial component of the Cauchy stress and

$$\Phi(v_1, v_2, v_3) = (v_1^2 + v_2^2 + v_3^2)^{p/2} + h(v_1 v_2 v_3). \tag{4.43}$$

It is a consequence of (4.41) that

$$\lim_{R \searrow 0} \frac{\tilde{r}_0'(R)\tilde{r}_0^2(R)}{R^2} = \lambda^3 a. \tag{4.44}$$

It is also easy to check that the stored energy function Φ given by (4.43) satisfies the conditions of Proposition 1.43, which ensures that

$$R \mapsto \frac{\tilde{r}_0'(R)\tilde{r}_0^2(R)}{R^2} \quad \text{is increasing on } (0,1]. \tag{4.45}$$

It follows from (4.41) and (4.45) that

$$\tilde{r}_0'(R)\tilde{r}_0^2(R) \geq \lambda^3 a R^2 \quad \text{for all } R \in (0,1]. \tag{4.46}$$

Let $r_0 : [0,1] \to [0,\infty)$ be such that $\tilde{r}_0 = \lambda r_0$. It follows from (4.46) that $r_0 \in \mathcal{C}_a$, where

$$\mathcal{C}_a := \left\{ r \in W^{1,1}(0,1) : r(1) = 1,\ r'r^2 \geq aR^2 \text{ a.e. on } (0,1], r(0) > 0 \right\}. \tag{4.47}$$

The convexity of h shows that

$$\begin{aligned} 0 > E(\tilde{\mathbf{u}}_0) - E(\mathbf{u}_{\lambda\mathbf{I}}^{\text{hom}}) &= \int_\Omega |\nabla\tilde{\mathbf{u}}_0|^2 + h(\det\nabla\tilde{\mathbf{u}}_0) - |\lambda\mathbf{I}|^2 - h(\lambda^3)\, d\mathbf{x} \\ &\geq \int_\Omega |\nabla\tilde{\mathbf{u}}_0|^2 - |\lambda\mathbf{I}|^2\, d\mathbf{x} + h'(\lambda^3)\int_\Omega (\det\nabla\tilde{\mathbf{u}}_0 - \det\lambda\mathbf{I})\, d\mathbf{x}. \end{aligned}$$

This implies that

$$\lambda \frac{\int_\Omega |\nabla\tilde{\mathbf{u}}_0|^2 - |\lambda\mathbf{I}|^2\, d\mathbf{x}}{\int_\Omega \det\lambda\mathbf{I} - \det\nabla\tilde{\mathbf{u}}_0\, d\mathbf{x}} < \frac{1}{2}(\gamma + 3), \tag{4.48}$$

and therefore, since

$$\lambda \frac{\int_\Omega |\nabla\tilde{\mathbf{u}}_0|^2 - |\lambda\mathbf{I}|^2\, d\mathbf{x}}{\int_\Omega \det\lambda\mathbf{I} - \det\nabla\tilde{\mathbf{u}}_0\, d\mathbf{x}} = \frac{\int_0^1 r_0'^2 R^2 + 2r_0^2\, dR - 1}{\frac{1}{3}r_0^3(0)},$$

it follows that
$$\frac{\int_0^1 r_0'^2 R^2 + 2r_0^2\, dR - 1}{\frac{1}{3}r_0^3(0)} < \frac{1}{2}(\gamma + 3). \tag{4.49}$$
Since $r_0 \in \mathcal{C}_a$, we clearly have
$$\frac{I(r_0) - 1}{\frac{1}{3}r_0^3(0)} \geq C_a, \tag{4.50}$$
where
$$C_a := \inf_{r \in \mathcal{C}_a} \frac{I(r) - 1}{\frac{1}{3}r^3(0)}$$
and, for every $r \in \mathcal{C}_a$, its energy I is given by (4.24). It follows from (4.49) that
$$C_a < \frac{1}{2}(\gamma + 3). \tag{4.51}$$
We shall determine explicitly the value of C_a, namely
$$C_a = 3\left(\frac{2p_a - 1}{p_a^2}\right), \text{ where } p_a \in [1, 3/2] \text{ satisfies } 3p_a^2 - 2p_a^3 = a. \tag{4.52}$$
Obviously, $\lim_{a \to 1} p_a = 1$ and
$$\lim_{a \to 1} C_a = \lim_{a \to 1} 3\frac{2p_a - 1}{p_a^2} = 3. \tag{4.53}$$
Since $\gamma < 3$ and (4.53) holds, it follows that (4.51) is contradicted for all a sufficiently close to 1, which would finish the proof, provided that (4.52) holds.

We now show that this is indeed the case. Given $a \in (0, 1)$ and $\alpha \in (0, 1)$, for $\mathcal{C}_{\alpha,a}$ to be non-empty it is necessary that (4.26) is satisfied. Let
$$C_{\alpha,a} = \inf_{r \in \mathcal{C}_{\alpha,a}} \frac{I(r) - 1}{\frac{1}{3}\alpha^3}. \tag{4.54}$$
Then, obviously,
$$C_a = \inf_{\alpha \in (0,1)} C_{\alpha,a}. \tag{4.55}$$
The minimiser of I on $\mathcal{C}_{\alpha,a}$ is $\tilde{r}_{\alpha,a}$ given by Lemma 4.7. A calculation using

(4.22) and (4.31) gives its energy as

$$
\begin{aligned}
I(\tilde{r}_{\alpha,a}) &= \int_0^A R^2\left[(r'_{\alpha,a})^2 + 2\left(\frac{r_{\alpha,a}}{R}\right)^2\right] dR + \int_A^1 R^2\left[(r'_{\alpha,a})^2 + 2\left(\frac{r_{\alpha,a}}{R}\right)^2\right] dR \\
&= -\frac{aA^4}{r_{\alpha,a}(A)} + 2Ar^2_{\alpha,a}(A) + \int_A^1 R^2\left[\left(c - \frac{d}{R^3}\right)^2 + 2\left(c + \frac{d}{2R^3}\right)^2\right] dR \\
&= A^3\left[2\left(c + \frac{d}{2A^3}\right)^2 - \left(c - \frac{d}{A^3}\right)\left(c + \frac{d}{2A^3}\right)\right] \\
&\qquad + c^2(1 - A^3) + \frac{d^2}{2}\left(\frac{1}{A^3} - 1\right) \\
&= A^3\left(c + \frac{d}{2A^3}\right)\left(c + \frac{2d}{A^3}\right) + c^2(1 - A^3) + \frac{d^2}{2}\left(\frac{1}{A^3} - 1\right).
\end{aligned} \tag{4.56}
$$

Also, from (4.33), (4.32) and (4.31), we obtain

$$
\begin{aligned}
\alpha^3 =& A^3\left[\left(c + \frac{d}{2A^3}\right)^3 - a\right] \\
=& A^3\left[\left(c + \frac{d}{2A^3}\right)^3 - \left(c + \frac{d}{2A^3}\right)^2\left(c - \frac{d}{A^3}\right)\right] \\
=& \frac{3d}{2}\left(c + \frac{d}{2A^3}\right)^2.
\end{aligned} \tag{4.57}
$$

It follows from (4.56), using (4.31c), that

$$
\begin{aligned}
I(\tilde{r}_{\alpha,a}) - 1 &= c^2A^3 + 2cd + \frac{cd}{2} + \frac{d^2}{A^3} + c^2 - c^2A^3 + \frac{d^2}{2A^3} - \frac{d^2}{2} - \left(c + \frac{d}{2}\right)^2 \\
&= \frac{d}{2}\left(3c + \frac{3d}{A^3} - \frac{3d}{2}\right) = \frac{3d}{2}\left[2\left(c + \frac{d}{2A^3}\right) - 1\right].
\end{aligned} \tag{4.58}
$$

Using (4.57) and (4.58), we obtain

$$
\frac{I(\tilde{r}_{\alpha,a}) - 1}{\frac{1}{3}\alpha^3} = 3\frac{2p_{\alpha,a} - 1}{p^2_{\alpha,a}}, \tag{4.59}
$$

where $p_{\alpha,a}$ is given by (4.29). It follows from (4.55), Lemma 4.7, and the fact that the mapping $p \mapsto 3p^2 - 2p^3$ is decreasing on the interval $[1, 3/2]$ that, if we denote by p_a the only solution of the equation $3p^2 - 2p^3 = a$, then

$$C_a = 3\left(\frac{2p_a - 1}{p_a^2}\right), \tag{4.60}$$

and this completes the proof of Theorem 4.6. $\square$

4.4 Necessary condition for $W^{1,2}$-quasiconvexity of W_α at a matrix A

Our next result gives a necessary condition for the stored energy density $W_\alpha(\mathbf{F}) = |\mathbf{F}|^2 + \alpha \det \mathbf{F}$ to be $W^{1,2}$-quasiconvex at a matrix $\mathbf{A}$ over $\mathcal{A}^*_{\mathbf{A},2}$.

Theorem 4.10. *If $W_\alpha(\mathbf{F}) = |\mathbf{F}|^2 + \alpha \det \mathbf{F}$ is $W^{1,2}$-quasiconvex at $\mathbf{A}$ over $\mathcal{A}^*_{\mathbf{A},2}$, then*

$$\alpha(\det \mathbf{A})^{1/3} \leq \Upsilon_2.$$

By Propositions 1.48 and 1.50, it suffices to consider the case of diagonal matrices $\mathbf{D}$ and of deformations in the unit ball B.

For general diagonal matrices $\mathbf{D}$, we now introduce a new class of deformations in B satisfying $\mathbf{u}(\mathbf{x}) = \mathbf{D}\mathbf{x}$ on ∂B, which provides a natural generalization of the class of radial deformations. (Note that radial deformations are possible only when $\mathbf{D} = \lambda\mathbf{I}$.) This new class will be used in the proof of Theorem 4.10, but one anticipates that it may find further use in nonlinear elasticity.

Let $\mathbf{D} = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)$ and let, for $i \in \{1, 2, 3\}$, $\mathbf{u}^{(i)}$ be radial deformations with $\mathbf{u}^{(i)}(\mathbf{x}) = \lambda_i\mathbf{x}$ on ∂B. We define $\mathcal{M}(\mathbf{u}^{(1)}, \mathbf{u}^{(2)}, \mathbf{u}^{(3)})$ to be the deformation $\mathbf{u} : B \to \mathbb{R}^3$ given by

$$\mathbf{u}(\mathbf{x}) = (u_1^{(1)}(\mathbf{x}), u_2^{(2)}(\mathbf{x}), u_3^{(3)}(\mathbf{x})). \tag{4.61}$$

In other words, if

$$\mathbf{u}^{(i)}(\mathbf{x}) = r^{(i)}(|\mathbf{x}|)\frac{\mathbf{x}}{|\mathbf{x}|} \quad \text{for } i \in \{1, 2, 3\},$$

then $\mathbf{u} = (u_1, u_2, u_3)$ is given by

$$u_i(\mathbf{x}) = r^{(i)}(|\mathbf{x}|)\frac{x_i}{|\mathbf{x}|} \quad \text{for } i \in \{1, 2, 3\}.$$

Note that $\mathbf{u}(\mathbf{x}) = \mathbf{D}\mathbf{x}$ on ∂B and, if $r^{(i)}(0) = \beta_i > 0$ for all $i \in \{1, 2, 3\}$, then $\mathbf{u}$ produces a hole at the origin, enclosed by the ellipsoid of equation

$$\frac{x_1^2}{\beta_1^2} + \frac{x_2^2}{\beta_2^2} + \frac{x_3^2}{\beta_3^2} = 1.$$

In fact the image under such a mapping of any sphere centred at the origin is an ellipsoid, see Figure 4-1.

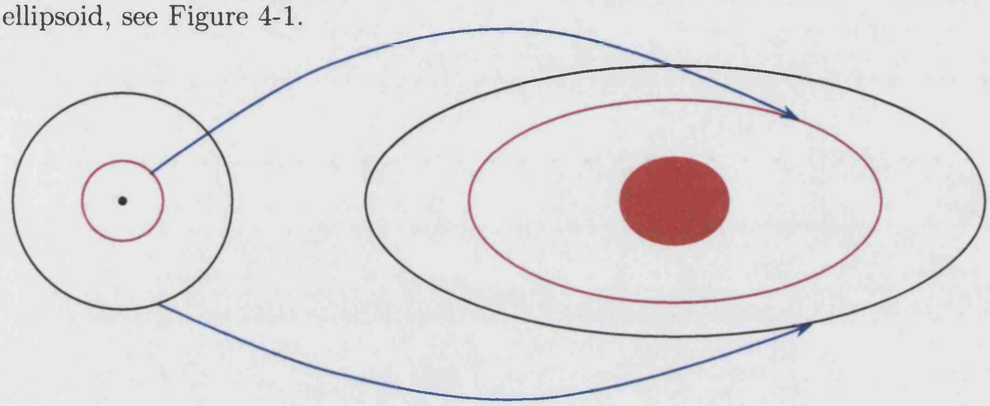

Figure 4-1: A deformation of the type $\mathcal{M}(\mathbf{u}^{(1)}, \mathbf{u}^{(2)}, \mathbf{u}^{(3)})$

Note also that, for such mappings $\mathbf{u}$,

$$\det \nabla\mathbf{u}(\mathbf{x}) = \frac{r^{(1)}r^{(2)}(r^{(3)})'}{|\mathbf{x}|^4}x_3^2 + \frac{r^{(1)}r^{(3)}(r^{(2)})'}{|\mathbf{x}|^4}x_2^2 + \frac{r^{(3)}r^{(2)}(r^{(1)})'}{|\mathbf{x}|^4}x_1^2, \tag{4.62}$$

so that $\det \nabla\mathbf{u} > 0$ almost everywhere whenever $(r^{(i)})' > 0$ almost everywhere for all $i \in \{1, 2, 3\}$. One can also see from (4.62) that

$$\int_B [\det \mathbf{D} - \det \nabla\mathbf{u}(\mathbf{x})]\, dx = \frac{4\pi}{3}\beta_1\beta_2\beta_3. \tag{4.63}$$

Proof of Theorem 4.10. The $W^{1,2}$-quasiconvexity of W_α shows, upon putting $\mathbf{u} =$

$\mathbf{Dv}$, that

$$\lambda_1^2 \int_B |\nabla v_1 - \mathbf{e}_1|^2 \, d\mathbf{x} + \lambda_2^2 \int_B |\nabla v_2 - \mathbf{e}_2|^2 \, d\mathbf{x} + \lambda_3^2 \int_B |\nabla v_3 - \mathbf{e}_3|^2 \, d\mathbf{x}$$
$$\geq \alpha\lambda_1\lambda_2\lambda_3 \int_B 1 - \det \nabla \mathbf{v} \, d\mathbf{x},$$

for all mappings $\mathbf{v} = (v_1, v_2, v_3)$ in $\mathcal{A}_{\mathrm{I},2}(\mathbf{0})$. Equivalently, for all such mappings $\mathbf{v}$,

$$\alpha(\lambda_1\lambda_2\lambda_3)^{1/3} \leq \frac{\lambda_1^2 \int_B |\nabla v_1 - \mathbf{e}_1|^2 \, d\mathbf{x} + \lambda_2^2 \int_B |\nabla v_2 - \mathbf{e}_2|^2 \, d\mathbf{x} + \lambda_3^2 \int_B |\nabla v_3 - \mathbf{e}_3|^2 \, d\mathbf{x}}{(\lambda_1\lambda_2\lambda_3)^{2/3} \int_B 1 - \det \nabla \mathbf{v} \, d\mathbf{x}}. \tag{4.64}$$

Now recall from Chapter 3 that, by (3.15),

$$\Upsilon_2 = \inf_{\beta\in(0,1)} \frac{\int_B |\nabla \mathbf{v}^{(\beta)} - \mathbf{I}|^2 \, d\mathbf{x}}{\frac{4\pi}{3}\beta^3} = \lim_{\beta\searrow 0} \frac{\int_B |\nabla \mathbf{v}^{(\beta)} - \mathbf{I}|^2 \, d\mathbf{x}}{\frac{4\pi}{3}\beta^3},$$

where $\mathbf{v}^{(\beta)} = (v_1^{(\beta)}, v_2^{(\beta)}, v_3^{(\beta)})$ is the radial deformation denoted by $\mathbf{u}_\beta$ in (3.15), opening a hole of radius β at the centre of the unit ball.

We now consider in (4.64) deformations of the type $\mathbf{v} = \mathcal{M}(\mathbf{v}^{(\beta_1)}, \mathbf{v}^{(\beta_2)}, \mathbf{v}^{(\beta_3)})$. This is allowed since, even if the mappings $\mathbf{v}^{(\beta)}$ do not belong to $\mathcal{A}_{\mathrm{I},2}(\mathbf{0})$, they can be approximated as in the proof of Theorem 3.8 by mappings in $\mathcal{A}_{\mathrm{I},2}(\mathbf{0})$, such that (3.48) holds with $p = 2$. We shall prove that, for every $\varepsilon > 0$, there exist $\beta_1, \beta_2, \beta_3$ such that the deformation $\mathbf{v} = \mathcal{M}(\mathbf{v}^{(\beta_1)}, \mathbf{v}^{(\beta_2)}, \mathbf{v}^{(\beta_3)})$ satisfies

$$\frac{\lambda_1^2 \int_B |\nabla v_1^{(\beta_1)} - \mathbf{e}_1|^2 \, d\mathbf{x} + \lambda_2^2 \int_B |\nabla v_2^{(\beta_2)} - \mathbf{e}_2|^2 \, d\mathbf{x} + \lambda_3^2 \int_B |\nabla v_3^{(\beta_3)} - \mathbf{e}_3|^2 \, d\mathbf{x}}{(\lambda_1\lambda_2\lambda_3)^{2/3} \int_B 1 - \det \nabla \mathbf{v} \, d\mathbf{x}}$$
$$\leq \Upsilon_2 + \varepsilon. \tag{4.65}$$

When combined with (4.64), (4.65) leads to the required conclusion.

It remains to prove that (4.65) indeed holds. Fix $\varepsilon > 0$. Then there exists β^* such that, for every $\beta \in (0, \beta^*)$ and $i \in \{1, 2, 3\}$,

$$\frac{\int_B |\nabla v_i^{(\beta)} - \mathbf{e}_i|^2 \, d\mathbf{x}}{\frac{4\pi}{3}\beta^3} \leq \frac{1}{3}\left(\Upsilon_2 + \varepsilon\right). \tag{4.66}$$

Let $\beta_1, \beta_2, \beta_3 \in (0, \beta^*)$ be such that

$$\lambda_1^2\beta_1^3 = \lambda_2^2\beta_2^3 = \lambda_3^2\beta_3^3, \tag{4.67}$$

and consider the corresponding deformation $\mathbf{v} = \mathcal{M}(\mathbf{v}^{(\beta_1)}, \mathbf{v}^{(\beta_2)}, \mathbf{v}^{(\beta_3)})$. It follows from (4.63), (4.66) and (4.67) that

$$\begin{aligned}
&\frac{\lambda_1^2 \int_B |\nabla v_1^{(\beta_1)} - \mathbf{e}_1|^2 \, d\mathbf{x} + \lambda_2^2 \int_B |\nabla v_2^{(\beta_2)} - \mathbf{e}_2|^2 \, d\mathbf{x} + \lambda_3^2 \int_B |\nabla v_3^{(\beta_3)} - \mathbf{e}_3|^2 \, d\mathbf{x}}{(\lambda_1\lambda_2\lambda_3)^{2/3} \int_B 1 - \det \nabla \mathbf{v} \, d\mathbf{x}} \\
&= \frac{\lambda_1^2 \int_B |\nabla v_1^{(\beta_1)} - \mathbf{e}_1|^2 \, d\mathbf{x} + \lambda_2^2 \int_B |\nabla v_2^{(\beta_2)} - \mathbf{e}_2|^2 \, d\mathbf{x} + \lambda_3^2 \int_B |\nabla v_3^{(\beta_3)} - \mathbf{e}_3|^2 \, d\mathbf{x}}{\frac{4\pi}{3}(\lambda_1\lambda_2\lambda_3)^{2/3}\beta_1\beta_2\beta_3} \\
&\le (\Upsilon_2 + \varepsilon)\frac{\lambda_1^2\beta_1^3 + \lambda_2^2\beta_2^3 + \lambda_3^2\beta_3^3}{3(\lambda_1\lambda_2\lambda_3)^{2/3}\beta_1\beta_2\beta_3} = \Upsilon_2 + \varepsilon.
\end{aligned} \tag{4.68}$$

Therefore (4.65) holds, and this completes the proof of Theorem 4.10. □

Chapter 5

On the Optimal Location of a Solitary Hole

In [44], Sivaloganathan and Spector showed that, for a large class of stored energy functions W, including W_h given by (1.29) with $2 < p < 3$, the associated energy has a minimiser in the class $\mathcal{A}_{\mathbf{A},p}(\mathbf{s})$ of mappings whose singularities can only occur at the point $\mathbf{s}$ in a domain Ω. By Proposition 1.49, the $W^{1,p}$-quasiconvexity of W over the class $\mathcal{A}_{\mathbf{A},p}(\mathbf{s})$ does not depend on the point $\mathbf{s}$ in Ω, and is equivalent to that over the class $\mathcal{A}^*_{\mathbf{A},p}$. We shall be interested in the situation when W is not $W^{1,p}$-quasiconvex over $\mathcal{A}^*_{\mathbf{A},p}$. In this case, it is an open problem of great interest whether the energy has a minimiser over the class $\mathcal{A}^*_{\mathbf{A},p}$. Equivalently, if $f : \Omega \to \mathbb{R}$ is given by

$$f(\mathbf{s}) := E(\mathbf{u_s}), \tag{5.1}$$

where $\mathbf{u_s}$ is a minimiser of the energy over $\mathcal{A}_{\mathbf{A},p}(\mathbf{s})$, it is not known whether f has a global minimum over Ω. The main difficulty is the apparent lack of an efficient method to compare the values of f at different points of Ω.

In this chapter we consider the case when $\mathbf{A} = \lambda\mathbf{I}$, Ω is the unit ball $B := B(\mathbf{0}, 1)$, and study the model energy function W_α given by

$$W_\alpha(\mathbf{F}) = |\mathbf{F}|^p + \alpha \det \mathbf{F} \quad \text{for all } \mathbf{F} \in M^{3\times 3}_+,$$

where $p \in [2, 3)$. Theorem 3.1 gives necessary and sufficient conditions on λ and α for the functional W_α to be $W^{1,p}$-quasiconvex over $\mathcal{A}^*_{\mathbf{A},p}$. But when W_α is not $W^{1,p}$-quasiconvex, no information is obtained on the existence of a minimiser over

$\mathcal{A}^*_{\mathbf{A},p}$ and its properties (except that it would have to be singular at some point). One would like to prove that such a minimiser exists and that it is radial. By scaling, it is enough to consider the case when $\lambda = 1$ and $\alpha > \Upsilon_p$.

Theorem 3.8 shows that the energy of any mapping having a singularity at $\mathbf{0}$ can be lowered by a mapping $\mathbf{u}_\beta$, for some $0 < \beta \leq 1$. One could ask whether the same is true for mappings which have a singularity at any other point in the domain. This is the question we are trying to answer in this chapter. We make the following conjecture.

Conjecture 5.1. *Let $\beta \in (0,1]$ and $\mathbf{s} \in B$. Let $\mathbf{u} \in \mathcal{A}_{\mathbf{I},p}(\mathbf{s})$ be such that*

$$\operatorname{Det} \nabla \mathbf{u} = (\det \nabla \mathbf{u})\mathcal{L}^3 + \frac{4\pi}{3}\beta^3 \delta_{\mathbf{s}}. \tag{5.2}$$

Let $\mathbf{u}_\beta = r_\beta(|\mathbf{x}|)\frac{\mathbf{x}}{|\mathbf{x}|}$ be as in Theorem 3.8. Then

$$\int_B |\nabla \mathbf{u}_\beta|^p \, d\mathbf{x} \leq \int_B |\nabla \mathbf{u}|^p \, d\mathbf{x}.$$

We suggest an approach to prove the following weaker version of Conjecture 5.1.

Conjecture 5.1'. *Let $\beta \in (0,1]$, let $\mathbf{u}_\beta = r_\beta(|\mathbf{x}|)\frac{\mathbf{x}}{|\mathbf{x}|}$ be as in Theorem 3.8, and let ε be such that $r_\beta(R) = \beta$ on $[0,\varepsilon]$. Let $\mathbf{s} \in B$ be such that $B(\mathbf{s},\varepsilon) \subset B$, and let $\mathbf{u} \in \mathcal{A}_{\mathbf{I},p}(\mathbf{s})$ be such that* (5.2) *holds. Then*

$$\int_B |\nabla \mathbf{u}_\beta|^p \, d\mathbf{x} \leq \int_B |\nabla \mathbf{u}|^p \, d\mathbf{x}.$$

To this aim, we make another conjecture, which we expect to be valid for any number $n \geq 2$ of space dimensions and any $p \geq 2$. Let $\varepsilon \in (0,1)$. For any $s \in [0, 1-\varepsilon)$, let $\mathbf{u}_{s,\varepsilon}$ be the unique minimizer of the p-energy on the annulus $B \setminus B((s,0'),\varepsilon)$ with Dirichlet boundary conditions $\mathbf{u}(\mathbf{x}) = \mathbf{x}$ on ∂B.

Conjecture 5.2. *For any $n \geq 2$ and $p \geq 2$,*

$$\int_{B\setminus B(\mathbf{0},\varepsilon)} |\nabla \mathbf{u}_{0,\varepsilon}|^p \, d\mathbf{x} \leq \int_{B\setminus B((s,0'),\varepsilon)} |\nabla \mathbf{u}_{s,\varepsilon}|^p \, d\mathbf{x} \quad \textit{for all } s \in [0,1-\varepsilon). \tag{5.3}$$

We now show that the validity of Conjecture 5.1' follows from the validity of

Conjecture 5.2. By the rotational invariance of the problem, it suffices to consider in Conjecture 5.1' only the case $\mathbf{s} = (s, \mathbf{0}')$.

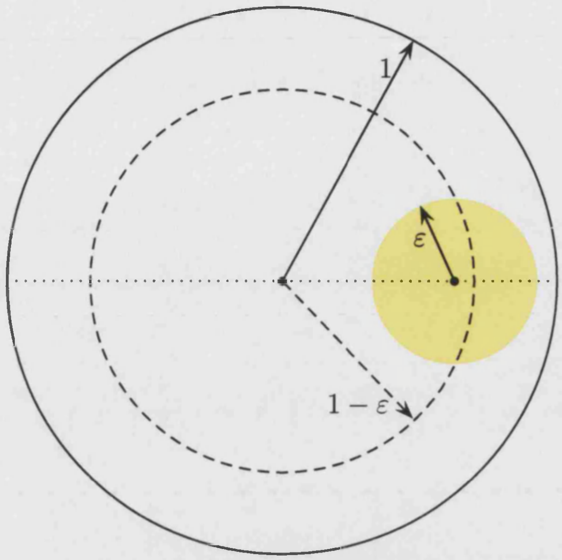

Figure 5-1: The relative position of the spheres

Then one can prove, by means of isoperimetric estimates exactly as in the proof of Theorem 3.8, the following result.

Proposition 5.3. *Let $\beta \in (0,1]$ and $\mathbf{s} = (s, \mathbf{0}')$ be such that $B(\mathbf{s}, \varepsilon) \subset B$, where ε is such that $r_\beta(R) = \beta$ on $[0, \varepsilon]$. Then, for any $\mathbf{u} \in \mathcal{A}_{\mathbf{I},p}(\mathbf{s})$ such that* (5.2) *holds,*

$$\int_{B(\mathbf{0},\varepsilon)} |\nabla \mathbf{u}_\beta|^p \, d\mathbf{x} \leq \int_{B(\mathbf{s},\varepsilon)} |\nabla \mathbf{u}|^p \, d\mathbf{x}.$$

Note that $\mathbf{u}_{0,\varepsilon}$ coincides with the restriction of $\mathbf{u}_\beta$ to the annulus $B \setminus B(\mathbf{0}, \varepsilon)$. Since, for any $\mathbf{u} \in W^{1,p}(B \setminus B(\mathbf{s}, \varepsilon))$,

$$\int_{B \setminus B(\mathbf{s},\varepsilon)} |\nabla \mathbf{u}|^p \, d\mathbf{x} \geq \int_{B \setminus B(\mathbf{s},\varepsilon)} |\nabla \mathbf{u}_{s,\varepsilon}|^p \, d\mathbf{x},$$

the proof of Conjecture 5.1' would be accomplished upon combining Conjecture 5.2 with Proposition 5.3.

In the remaining part of this chapter we rigorously prove Conjecture 5.2 in the case where $n = 2$ and $p = 2$. Unfortunately, the current method relies heavily on conformal mappings and cannot obviously be extended to higher dimensions or exponents $p \neq 2$.

For any $\varepsilon \in (0,1)$ and $s \in [0, 1-\varepsilon)$, let $\mathbf{u}_{s,\varepsilon}$ be the function defined on $B(\mathbf{0},1) \setminus B((s,0),\varepsilon)$ which is harmonic on this domain, satisfies zero Neumann boundary condition on $\partial B(\mathbf{0},\varepsilon)$ and $\mathbf{u}_{s,\varepsilon}(\mathbf{x}) = \mathbf{x}$ on $\partial B(\mathbf{0},1)$. For fixed $\varepsilon \in (0,1)$, we are interested to compare the Dirichlet energy of $\mathbf{u}_{s,\varepsilon}$ as s varies in $[0, 1-\varepsilon)$, and to show that it is minimal when $s = 0$. From now on, we denote $\mathbf{s} := (s, 0)$, for $s \in [0, 1-\varepsilon)$.

Theorem 5.4. *Let $\varepsilon \in (0,1)$. Then*

$$E(\mathbf{u}_{0,\varepsilon}) < E(\mathbf{u}_{s,\varepsilon}) \quad \textit{for all } s \in (0, 1-\varepsilon),$$

where

$$E(\mathbf{u}_{s,\varepsilon}) := \int_{B\setminus B(\mathbf{s},\varepsilon)} |\nabla \mathbf{u}_{s,\varepsilon}|^2 d\mathbf{x} \quad \textit{for all } s \in [0, 1-\varepsilon).$$

We prove this theorem by relating $E(\mathbf{u}_{s,\varepsilon})$ to the Dirichlet energy of a harmonic function $\mathbf{v}_{s,\varepsilon}$ on an annulus $B \setminus B(\mathbf{0},\rho)$ for some suitable $\rho \in (0,1)$. We start with a discussion on conformal mappings.

It is well known, see [38, Theorem 12.4], that for every $t \in (-1,1)$ the mapping $\varphi_t : \overline{B}(\mathbf{0},1) \to \overline{B}(\mathbf{0},1)$ given by

$$\varphi_t(z) = \frac{z+t}{1+tz} \qquad \text{for all } z \in \overline{B}(\mathbf{0},1),$$

is a conformal mapping from B onto B, and a homeomorphism from $\overline{B}$ onto $\overline{B}$, with inverse given by φ_{-t}.

Proposition 5.5. *Let $t \in (-1,1)$ and $\rho \in (0,1)$. Then the image $\varphi_t(S(\mathbf{0},\rho))$ is a circle $S(\mathrm{s},\varepsilon)$ contained in $B(\mathbf{0},1)$, with the centre and radius given by*

$$s = \frac{t(1-\rho^2)}{1-\rho^2 t^2}, \tag{5.4a}$$

$$\varepsilon = \frac{\rho(1-t^2)}{1-\rho^2 t^2}. \tag{5.4b}$$

Proof of Proposition 5.5. Let $w = \varphi_t(z)$, where $z \in S(\mathbf{0},\rho)$. Then $z = \varphi_{-t}(w)$, and we have the following equivalences

$$
\begin{aligned}
|z| = \rho &\iff |w-t|^2 = \rho^2|1-tw|^2 \\
&\iff (1-\rho^2t^2)|w|^2 - t(1-\rho^2)(w+\overline{w}) = \rho^2 - t^2 \\
&\iff |w|^2 - \frac{t(1-\rho^2)}{1-\rho^2t^2}(w+\overline{w}) = \frac{\rho^2-t^2}{1-\rho^2t^2} \\
&\iff \left|w - \frac{t(1-\rho^2)}{1-\rho^2t^2}\right|^2 = \frac{t^2(1-\rho^2)^2}{(1-\rho^2t^2)^2} + \frac{\rho^2-t^2}{1-\rho^2t^2} \\
&\iff \left|w - \frac{t(1-\rho^2)}{1-\rho^2t^2}\right|^2 = \frac{\rho^2(1-t^2)^2}{(1-\rho^2t^2)^2}.
\end{aligned}
$$

Hence indeed $\varphi_t(S(\mathbf{0},\rho)) = S(\mathbf{s},\varepsilon)$, where s and ε are given by (5.4). $\square$

We now show that, conversely, for any $s \in (-1,1)$ and $\varepsilon > 0$ such that $S(\mathbf{s},\varepsilon) \subseteq B(\mathbf{0},1)$, there exist unique $t \in (-1,1)$ and $\rho \in (0,1)$ such that $S(\mathbf{s},\varepsilon) = \varphi_t(S(\mathbf{0},\rho))$.

Proposition 5.6. *For any $s \in (-1,1)$ and $\varepsilon > 0$ such that*

$$-1 < s-\varepsilon < s+\varepsilon < 1, \tag{5.5}$$

there exists a unique solution $(t,\rho) \in (-1,1)\times(0,1)$ of (5.4). Moreover, this solution (t,ρ) coincides with the unique solution in $(-1,1)\times(0,1)$ of the quadratic equations

$$\frac{2t}{t^2+1} = \frac{2s}{1+s^2-\varepsilon^2}, \tag{5.6a}$$
$$\frac{2\rho}{\rho^2+1} = \frac{2\varepsilon}{1+\varepsilon^2-s^2}. \tag{5.6b}$$

Proof of Proposition 5.6 . Note first that, indeed, when (5.5) holds, (5.6a) has a unique solution in $(-1,1)$ and (5.6b) has a unique solution in $(0,1)$.

We now want to show that (5.4) and (5.6) are equivalent for $s \in (-1,1)$ and $\varepsilon > 0$ satisfying (5.5), and $t \in (-1,1)$ and $\rho \in (0,1)$.

Let $a := s+\varepsilon$, $b := s-\varepsilon$, so it follows from (5.5) that $-1 < b < a < 1$. Then

(5.4) can be rewritten as

$$a = \frac{t+\rho}{1+t\rho}, \tag{5.7a}$$

$$b = \frac{t-\rho}{1-t\rho}, \tag{5.7b}$$

while (5.6) can be rewritten as

$$\frac{2t}{t^2+1} = \frac{a+b}{1+ab}, \tag{5.8a}$$

$$\frac{2\rho}{\rho^2+1} = \frac{a-b}{1-ab}. \tag{5.8b}$$

Recall now that the function $\tanh : \mathbb{R} \to (-1,1)$ given by

$$\tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

is a homeomorphism from $\mathbb{R}$ onto $(-1,1)$, and satisfies the formula

$$\tanh(x+y) = \frac{\tanh(x) + \tanh(y)}{1 + \tanh(x)\tanh(y)} \quad \text{for all } x, y \in \mathbb{R}. \tag{5.9}$$

Let $f_a, f_b, f_t, f_\rho \in \mathbb{R}$ be such that

$$\tanh(f_z) = z \quad \text{for all } z \in \{a, b, t, \rho\}.$$

Then, in view of (5.9), (5.7) can be rewritten as

$$f_a = f_t + f_\rho, \tag{5.10a}$$

$$f_b = f_t - f_\rho, \tag{5.10b}$$

where $f_a > f_b$, while (5.8) can be rewritten as

$$2f_t = f_a + f_b, \tag{5.11a}$$

$$2f_\rho = f_a - f_b, \tag{5.11b}$$

where $f_\rho > 0$.

Since (5.10) and (5.11) are obviously equivalent, it follows that (5.7) and (5.8) are equivalent, and therefore (5.4) and (5.6) are equivalent. This completes the proof. $\square$

Remark 5.7. Note that in (5.4), or equivalently in (5.6), $s > 0$ if and only if $t > 0$.

Proof of Theorem 5.4. Let $\varepsilon > 0$ and $s > 0$ be such that $S(\mathbf{s}, \varepsilon) \subseteq B(\mathbf{0}, 1)$. Let $t \in (0,1)$ and $\rho \in (0,1)$ be such that $S(\mathbf{s}, \varepsilon) = \varphi_t(S(\mathbf{0}, \rho))$. Then one can easily check that $\overline{B}(\mathbf{0}, 1) \setminus B(\mathbf{s}, \varepsilon) = \varphi_t(\overline{B}(\mathbf{0}, 1) \setminus B(\mathbf{0}, \rho))$.

Let $\mathbf{v}_{s,\varepsilon} : \overline{B}(\mathbf{0}, 1) \setminus B(\mathbf{0}, \rho) \to \mathbb{R}^2$ be given by

$$\mathbf{v}_{s,\varepsilon} = \mathbf{u}_{s,\varepsilon} \circ \varphi_t. \tag{5.12}$$

For convenience of notation, in what follows we write $\mathbf{u}$ and $\mathbf{v}$ instead of $\mathbf{u}_{s,\varepsilon}$ and $\mathbf{v}_{s,\varepsilon}$.

Let $\Omega = B(\mathbf{0}, 1) \setminus B(\mathbf{s}, \varepsilon)$ and $\widetilde{\Omega} = B(\mathbf{0}, 1) \setminus B(\mathbf{0}, \rho)$. Then it is easy to check, since φ_t is a conformal mapping, that $\mathbf{v}$ satisfies

$$\Delta \mathbf{v} = 0 \quad \text{on } \widetilde{\Omega}, \tag{5.13a}$$

$$\mathbf{v} = \varphi_t \quad \text{on } S(\mathbf{0}, 1), \tag{5.13b}$$

$$\frac{\partial \mathbf{v}}{\partial \mathbf{n}} = 0 \quad \text{on } S(\mathbf{0}, \rho), \tag{5.13c}$$

and that

$$\int_{\Omega} |\nabla \mathbf{u}(\mathbf{x})|^2 \, d\mathbf{x} = \int_{\widetilde{\Omega}} |\nabla \mathbf{v}(\mathbf{y})|^2 \, d\mathbf{y}. \tag{5.14}$$

Note that, for every $z \in \overline{B}(\mathbf{0}, 1)$, the following holds

$$\begin{aligned} \varphi_t(z) &= \frac{z+t}{1+tz} = (z+t) \sum_{n=0}^{\infty} (-tz)^n \\ &= t + \sum_{n=1}^{\infty} (-1)^{n-1} (t^{n-1} - t^{n+1}) z^n. \end{aligned} \tag{5.15}$$

In particular, for $z = e^{i\theta}$, $\theta \in \mathbb{R}$,

$$\varphi_t(z) = t + (1 - t^2) \sum_{n \geq 1} (-1)^{n-1} t^{n-1} (\cos n\theta + i \sin n\theta).$$

It is well known that any harmonic function $f : \widetilde{\Omega} \to \mathbb{R}$ has, in polar coordinates, an expansion of the form

$$f(re^{i\theta}) = c\log r + d + \sum_{n\geq 1}(a_n r^n + a_{-n}r^{-n})\cos n\theta + \sum_{n\geq 1}(b_n r^n + b_{-n}r^{-n})\sin n\theta,$$
$$\text{for all } r \in (\rho, 1),\ \theta \in \mathbb{R}.$$

It follows that, for all $\theta \in \mathbb{R}$,

$$f(e^{i\theta}) = d + \sum_{n\geq 1}(a_n + a_{-n})\cos n\theta + \sum_{n\geq 1}(b_n + b_{-n})\sin n\theta,$$

and

$$\begin{aligned}\frac{\partial f}{\partial r}(re^{i\theta})\Big|_{r=\rho} = \frac{c}{\rho} &+ \sum_{n\geq 1}(na_n\rho^{n-1} - na_{-n}\rho^{-n-1})\cos n\theta \\ &+ \sum_{n\geq 1}(nb_n\rho^{n-1} - nb_{-n}\rho^{-n-1})\sin n\theta.\end{aligned}$$

Since $\mathbf{v} = (v_1, v_2)$ satisfies (5.13), it follows from the above relations that, for all $r \in [\rho, 1]$, $\theta \in \mathbb{R}$,

$$v_1(re^{i\theta}) = t + (1-t^2)\sum_{n\geq 1}(-1)^{n-1}t^{n-1}\frac{1}{1+\rho^{2n}}\left(r^n + \frac{\rho^{2n}}{r^n}\right)\cos n\theta,$$
$$v_2(re^{i\theta}) = (1-t^2)\sum_{n\geq 1}(-1)^{n-1}t^{n-1}\frac{1}{1+\rho^{2n}}\left(r^n + \frac{\rho^{2n}}{r^n}\right)\sin n\theta.$$

and also that

$$\frac{\partial v_1}{\partial r}(re^{i\theta}) = (1-t^2)\sum_{n\geq 1}(-1)^{n-1}t^{n-1}\frac{n}{1+\rho^{2n}}\left(r^{n-1} - \frac{\rho^{2n}}{r^{n+1}}\right)\cos n\theta,$$

$$\frac{\partial v_2}{\partial r}(re^{i\theta}) = (1-t^2)\sum_{n\geq 1}(-1)^{n-1}t^{n-1}\frac{n}{1+\rho^{2n}}\left(r^{n-1} - \frac{\rho^{2n}}{r^{n+1}}\right)\sin n\theta.$$

It follows from Green's formula that

$$\int_{\tilde{\Omega}} |\nabla \mathbf{v}|^2 \, d\mathbf{y} = \int_{\tilde{\Omega}} |\nabla v_1|^2 + |\nabla v_2|^2 \, d\mathbf{y}$$
$$= \int_{S(0,1)} \left(v_1 \frac{\partial v_1}{\partial r} + v_2 \frac{\partial v_2}{\partial r} \right) \, d\mathcal{H}^1. \tag{5.16}$$

We obtain, upon using Parseval's formula (i.e. the orthogonality in L^2 of the trigonometric system), that

$$\int_{\tilde{\Omega}} |\nabla \mathbf{v}|^2 \, d\mathbf{y} = 2\pi \sum_{n\geq 1} n t^{2n-2} (1-t^2)^2 \frac{1-\rho^{2n}}{1+\rho^{2n}}.$$

Using (5.14), we deduce that

$$E(\mathbf{u}_{s,\varepsilon}) = 2\pi \sum_{n\geq 1} n t^{2n-2} (1-t^2)^2 \frac{1-\rho^{2n}}{1+\rho^{2n}},$$

where $t \in (0,1)$, $\rho \in (0,1)$ are related to s and ε by (5.6). In particular, when $s = 0$, one can see that $t = 0$, $\rho = \varepsilon$, so that

$$E(\mathbf{u}_{0,\varepsilon}) = 2\pi \, \frac{1-\varepsilon^2}{1+\varepsilon^2}.$$

We need to prove that

$$\frac{1-\varepsilon^2}{1+\varepsilon^2} < A, \tag{5.17}$$

where A is such that $E(\mathbf{u}_{s,\varepsilon}) = 2\pi A$, namely

$$A := \sum_{n\geq 1} n t^{2n-2} (1-t^2)^2 \frac{1-\rho^{2n}}{1+\rho^{2n}}.$$

Rearranging (5.17) gives

$$\frac{1-A}{1+A} < \varepsilon^2. \tag{5.18}$$

Note that, by (5.4),

$$\varepsilon^2 = \frac{\rho^2(1-t^2)^2}{(1-t^2\rho^2)^2},$$

so it remains to prove that

$$\frac{1-A}{1+A} < \frac{\rho^2(1-t^2)^2}{(1-t^2\rho^2)^2}, \tag{5.19}$$

for every $(t,\rho) \in (0,1) \times (0,1)$. Upon denoting by $x := t^2$, $y := \rho^2$, $x, y \in (0,1)$, proving (5.19) is equivalent to proving

$$\frac{1-\sum_{n\geq 1} nx^{n-1}(1-x)^2\frac{1-y^n}{1+y^n}}{1+\sum_{n\geq 1} nx^{n-1}(1-x)^2\frac{1-y^n}{1+y^n}} < \frac{y(1-x)^2}{(1-xy)^2}. \tag{5.20}$$

Using now the identity

$$\sum_{n\geq 1} nz^{n-1}(1-z)^2 = 1 \text{ for all } z \in (0,1), \tag{5.21}$$

(5.20) can be written as

$$\frac{\sum_{n\geq 1} nx^{n-1}(1-x)^2\left(1-\frac{1-y^n}{1+y^n}\right)}{\sum_{n\geq 1} nx^{n-1}(1-x)^2\left(1+\frac{1-y^n}{1+y^n}\right)} < \frac{y(1-x)^2}{(1-xy)^2}. \tag{5.22}$$

This is equivalent to

$$\frac{y(1-x)^2\sum_{n\geq 1} nx^{n-1}\frac{y^{n-1}}{1+y^n}}{\sum_{n\geq 1} nx^{n-1}(1-x)^2\frac{1}{1+y^n}} < \frac{y(1-x)^2}{(1-xy)^2},$$

which can be rewritten as

$$\sum_{n\geq 1} n(xy)^{n-1}(1-xy)^2\frac{1}{1+y^n} < \sum_{n\geq 1} nx^{n-1}(1-x)^2\frac{1}{1+y^n}. \tag{5.23}$$

Let, for all $n \geq 1$, $a_n := n(xy)^{n-1}(1-xy)^2$, $b_n := nx^{n-1}(1-x)^2$, $c_n := \frac{1}{1+y^n}$. Since $y \in (0,1)$, the sequence $\{c_n\}_{n\geq 1}$ is strictly increasing. Also, (5.21) shows that

$$\sum_{n\geq 1} a_n = \sum_{n\geq 1} b_n = 1. \tag{5.24}$$

It is easy to see that there exists $N \in \mathbb{N}$ such that

$$a_n \geq b_n \quad \text{if and only if } n \leq N. \tag{5.25}$$

We now write (5.23) as

$$\sum_{1 \leq n \leq N} (a_n - b_n)c_n < \sum_{n>N} (b_n - a_n)c_n. \tag{5.26}$$

Note that, by (5.24),

$$\sum_{1 \leq n \leq N} (a_n - b_n) = \sum_{n>N} (b_n - a_n) := S > 0. \tag{5.27}$$

Since $\{c_n\}_{n \geq 1}$ is increasing, we obtain that

$$\sum_{1 \leq n \leq N} (a_n - b_n)c_n \leq Sc_N \tag{5.28}$$

and

$$\sum_{n>N} (b_n - a_n)c_n > Sc_N. \tag{5.29}$$

Combining the relations (5.28) and (5.29), we get that (5.26) holds, which means that (5.18) holds. The proof that

$$E(\mathbf{u}_{0,\varepsilon}) < E(\mathbf{u}_{s,\varepsilon}) \quad \text{for all } \varepsilon > 0, \ s \in (0, 1 - \varepsilon)$$

is therefore completed. □

Chapter 6

On the Equilibrium Equations of Nonlinear Elasticity

In this chapter we study solutions of a certain equilibrium equation of nonlinear elasticity, namely the weak form of the Green Divergence Identity

$$-\int_\Omega \frac{\partial \phi}{\partial x^\alpha}\left[x^\alpha W(\nabla \mathbf{u}) + \left(u^i - \frac{\partial u^i}{\partial x^k}x^k\right)\frac{\partial W}{\partial F^i_\alpha}(\nabla \mathbf{u})\right] d\mathbf{x} = 3\int_\Omega W(\nabla \mathbf{u})\phi \, d\mathbf{x}, \tag{6.1}$$

$$\text{for all } \phi \in C^1_0(\Omega; \mathbb{R}).$$

This equation is of interest since its strong form

$$\frac{\partial}{\partial x^\alpha}\left[x^\alpha W(\nabla \mathbf{u}) + \left(u^i - \frac{\partial u^i}{\partial x^k}x^k\right)\frac{\partial W}{\partial F^i_\alpha}(\nabla \mathbf{u})\right] = 3W(\nabla \mathbf{u}) \tag{6.2}$$

is satisfied by any C^2 solution of (6.3) (see Green [20] and Knops and Stuart [28]).

In Section 6.1 we study the uniqueness of solutions of (6.1) under affine boundary displacements. In Section 6.2 we derive (6.1) as a necessary condition for a local extremum of the energy functional.

It is assumed throughout the chapter that Ω is a star-shaped domain with respect to the origin, with a boundary of class C^1.

6.1 Uniqueness of solutions of the weak form of the Green Divergence Identity

The uniqueness of classical solutions, satisfying affine boundary displacements, i.e. $\mathbf{u}(\mathbf{x}) = \mathbf{A}\mathbf{x}$ for all $\mathbf{x} \in \partial\Omega$, where $\mathbf{A} \in M_+^{3\times3}$, of the Euler-Lagrange equations

$$\frac{\partial}{\partial x^\alpha}\left[\frac{\partial W}{\partial F_\alpha^i}(\nabla\mathbf{u}(\mathbf{x}))\right] = 0, \quad \text{for i } = 1, 2, 3, \tag{6.3}$$

was investigated by Knops and Stuart [28]. They showed that if the stored energy function W is of class C^2 on $M_+^{3\times3}$, rank-one convex on $M_+^{3\times3}$ and strictly quasiconvex at $\mathbf{A}$, then the only solution $\mathbf{u} \in C^2(\Omega; \mathbb{R}^3) \cap C^1(\overline{\Omega}; \mathbb{R}^3)$, satisfying $\det \nabla\mathbf{u} > 0$ in $\overline{\Omega}$, of (6.3) is the affine mapping $\mathbf{u}_{\mathbf{A}}^{\text{hom}}$.

Definition 6.1. *The stored energy function W is said to be* **rank-one-convex at F**, *where* $\mathbf{F} \in M_+^{3\times3}$, *if*

$$W(\mathbf{F} + \mu\mathbf{a}\otimes\mathbf{b}) \leq \mu W(\mathbf{F} + \mathbf{a}\otimes\mathbf{b}) + (1-\mu)W(\mathbf{F}) \quad \textit{for all } \mu \in [0,1],$$

whenever $\mathbf{a} \in \mathbb{R}^3$, $\mathbf{b} \in \mathbb{R}^3$ *are such that* $\mathbf{F} + t\mathbf{a}\otimes\mathbf{b} \in M_+^{3\times3}$ *for all* $t \in [0,1]$.

We say that W is **rank-one-convex on** $M_+^{3\times3}$ *if W is rank-one-convex at* $\mathbf{F}$ *for all* $\mathbf{F} \in M_+^{3\times3}$.

In their result, the requirement that $\mathbf{u}$ is a classical solution of (6.3) is restrictive, since in most situations the global (or local) minimisers of the energy associated to W are only known to lie in a Sobolev space $W^{1,p}(\Omega; \mathbb{R}^3)$, and one can expect that they satisfy only the weak form of (6.3), i.e.

$$\int_\Omega \frac{\partial W}{\partial F}(\nabla\mathbf{u}(\mathbf{x})) : \nabla\varphi(\mathbf{x})\, d\mathbf{x} = 0 \quad \text{for all } \varphi \in C_0^1(\Omega; \mathbb{R}^3). \tag{6.4}$$

In fact, it is an open problem whether the minimisers obtained by Ball [4] and others satisfy (6.4). Other forms of equilibrium equations have been considered in the literature, for which it is possible to show that they are satisfied by minimisers. One such set of equations is the energy-momentum equations, the weak form of which is

$$\int_\Omega \left[W(\nabla \mathbf{u}(\mathbf{x}))\mathbf{I} - (\nabla \mathbf{u}(\mathbf{x}))^T \frac{\partial W}{\partial F}(\nabla \mathbf{u}(\mathbf{x}))\right] : \nabla \varphi(\mathbf{x})\, d\mathbf{x} = 0,$$
$$\text{for all } \varphi \in C_0^1(\Omega; \mathbb{R}^3). \qquad (6.5)$$

Recently, Taheri [51] showed that if W is of class C^1 on $M^{3\times 3}$ and satisfies suitable growth conditions, then the only solution $\mathbf{u}$ in $W^{1,p}(\Omega; \mathbb{R}^3)$ of (6.4) and (6.5) which is of class C^1 near $\partial\Omega$ is the affine mapping $\mathbf{u}_\mathbf{A}^{\text{hom}}$. The requirement that $\mathbf{u}$ is C^1 near $\partial\Omega$, although it does not occur explicitly in [51], is implicit in order to have a meaningful result (see [51, Theorem 2.1]). Although the growth conditions in [51] are incompatible with nonlinear elasticity, a result which applies to elasticity can easily be deduced. This result is stronger than that in [28] since C^2 solutions of (6.3) satisfy both (6.4) and (6.5). The uniqueness result in [28] has also been recovered by Sivaloganathan in [40] by a different method, using one-parameter families of symmetry transformations.

In the uniqueness result in [28], an essential role is played by the fact that classical solutions of the Euler-Lagrange equation satisfy the Green Divergence Identity (6.2). Here we consider the uniqueness of solutions of (6.1), the weak form this identity.

Remark 6.2. The radial form of (6.2), mentioned in Chapter 1 (Proposition 1.44 and Remark 1.45), was also used in Chapter 3 (see the proof of the Theorem 3.9).

We assume henceforth that $\mathbf{u} \in W^{1,1}(\Omega; \mathbb{R}^3)$ with $E(\mathbf{u}) < \infty$ has additional smoothness near $\partial\Omega$, namely that there exists $\tau \in [0, 1)$ such that

$$\mathbf{u} \in C^1(\overline{\Omega} \setminus \tau\Omega) \qquad \text{and} \qquad \det \nabla \mathbf{u} > 0 \text{ in } \overline{\Omega} \setminus \tau\Omega, \qquad (6.6)$$

where $\tau\Omega := \{\tau\mathbf{x} : \mathbf{x} \in \Omega\}$. Note that the condition $E(\mathbf{u}) < \infty$ implies that $\det \nabla \mathbf{u} > 0$ almost everywhere in Ω, while (6.6) shows that there exists $\varepsilon > 0$ such that

$$\det \nabla \mathbf{u} \geq \varepsilon \quad \text{in } \overline{\Omega} \setminus \tau\Omega.$$

Note also that the condition that $E(\mathbf{u}) < \infty$ need not imply the integrability of the left-hand side in (6.1) and hence, since we do not wish to impose any extra condition on $\dfrac{\partial W}{\partial \mathbf{F}}(\nabla \mathbf{u})$, we restrict the class of functions ϕ in (6.1) to those for which

$$\phi|_{\mathcal{T}\Omega} = \text{constant}. \tag{6.7}$$

Hence we consider solutions of

$$-\int_\Omega \frac{\partial\phi}{\partial x^\alpha}\left[x^\alpha W(\nabla\mathbf{u}) + \left(u^i - \frac{\partial u^i}{\partial x^k}x^k\right)\frac{\partial W}{\partial F^i_\alpha}(\nabla\mathbf{u})\right]\,d\mathbf{x} = 3\int_\Omega W(\nabla\mathbf{u})\phi\,d\mathbf{x}, \tag{6.8}$$

for all $\phi \in C^1_0(\Omega;\mathbb{R})$ satisfying (6.7).

Theorem 6.3. *Let* $\mathbf{A} \in M^{3\times3}_+$ *and let* $\mathbf{u} \in W^{1,1}(\Omega;\mathbb{R}^3)$ *satisfying* (6.6), $E(\mathbf{u}) < \infty$ *and* $\mathbf{u}(\mathbf{x}) = \mathbf{A}\mathbf{x}$ *on* $\partial\Omega$, *be such that* (6.8) *holds. If* W *is rank-one convex on* $M^{3\times3}_+$, *then* $E(\mathbf{u}) \le E(\mathbf{u}^{\mathrm{hom}}_{\mathbf{A}})$. *Hence if* W *is strictly* $W^{1,1}$*-quasiconvex at* $\mathbf{A}$, *then* $\mathbf{u}(\mathbf{x}) = \mathbf{A}\mathbf{x}$ *for all* $\mathbf{x} \in \Omega$.

The following lemma and corollary, taken from [28], are straightforward to prove and will be used in the proof of Theorem 6.3.

Lemma 6.4. ([28, Lemma 2.1]) *Let* $\mathbf{u}, \mathbf{v} \in C^1(\overline{\Omega};\mathbb{R}^3)$ *with* $\mathbf{u}(\mathbf{x}) = \mathbf{v}(\mathbf{x})$ *for all* $\mathbf{x} \in \partial\Omega$. *Then*

(i) $\nabla(\mathbf{u}(\mathbf{x}) - \mathbf{v}(\mathbf{x})) = \dfrac{\partial}{\partial N}\Big(\mathbf{u}(\mathbf{x}) - \mathbf{v}(\mathbf{x})\Big) \otimes N(\mathbf{x}), \qquad \mathbf{x} \in \partial\Omega,$

(ii) $\nabla(\mathbf{u}(\mathbf{x}) - \mathbf{v}(\mathbf{x}))\mathbf{x} = (\mathbf{x}\cdot N(\mathbf{x}))\dfrac{\partial}{\partial N}\Big(\mathbf{u}(\mathbf{x}) - \mathbf{v}(\mathbf{x})\Big), \qquad \mathbf{x} \in \partial\Omega,$

where $N(\mathbf{x})$ *is the outward unit normal to* $\partial\Omega$ *at the point* $\mathbf{x} \in \partial\Omega$.

Corollary 6.5. *Let* $\mathbf{u} \in C^1(\overline{\Omega};\mathbb{R}^3)$. *Then*

$$[(\nabla\mathbf{u}(\mathbf{x}) - \nabla\mathbf{u}^{\mathrm{hom}}_{\mathbf{A}}(\mathbf{x}))\mathbf{x}]\otimes N(\mathbf{x}) = (\nabla\mathbf{u}(\mathbf{x}) - \nabla\mathbf{u}^{\mathrm{hom}}_{\mathbf{A}}(\mathbf{x}))(\mathbf{x}\cdot N(\mathbf{x})) \quad \textit{for all } \mathbf{x} \in \partial\Omega.$$

Proof of Theorem 6.3. Since Ω is a star-shaped domain with respect to the origin and $\partial\Omega$ is of class C^1, there exists a C^1 function $d : \mathcal{S}^2 \to (0,\infty)$, where $\mathcal{S}^2$ is the unit sphere in $\mathbb{R}^3$, such that

$$\Omega = \{\mathbf{0}\} \cup \{\mathbf{x} \in \mathbb{R}^3 \setminus \{\mathbf{0}\} : |\mathbf{x}| < d(\theta)\} \tag{6.9}$$

$$\partial\Omega = \{\mathbf{x} \in \mathbb{R}^3 : |\mathbf{x}| = d(\theta)\}, \tag{6.10}$$

where $\theta = \mathbf{x}/|\mathbf{x}|$. The unit outer normal to $\partial\Omega$ is then given (see [51]) by

$$N(\mathbf{x}) = \frac{1}{\alpha(\theta)}\left(\theta - (\mathbf{I} - \theta\otimes\theta)\frac{\nabla d(\theta)}{d(\theta)}\right), \tag{6.11}$$

where

$$\alpha(\theta) = \frac{1}{d(\theta)}\left(d(\theta)^2 + |\nabla d(\theta)|^2 - (\theta \cdot \nabla d(\theta))^2\right)^{1/2}. \tag{6.12}$$

The following formula (see [51]) will be useful: for every $f \in L^1(\Omega)$,

$$\int_\Omega f(\mathbf{x})\,d\mathbf{x} = \int_0^1 \rho^2 \int_{\partial\Omega} \frac{d(\theta)}{\alpha(\theta)} f(\rho\mathbf{x})\,d\mathcal{H}^2(\mathbf{x})\,d\rho. \tag{6.13}$$

Let now $\mathbf{u}$ be as in the statement of the theorem. Note that (6.8) can be equivalently rewritten as

$$\begin{aligned} 3\int_\Omega W(\nabla\mathbf{u}(\mathbf{x}))\phi(\mathbf{x})\,d\mathbf{x} = &- \int_\Omega W(\nabla\mathbf{u}(\mathbf{x}))(\nabla\phi(\mathbf{x})\cdot\mathbf{x})\,d\mathbf{x} \\ &- \int_\Omega \frac{\partial W}{\partial \mathbf{F}}(\nabla\mathbf{u}(\mathbf{x})) : [(\mathbf{u}(\mathbf{x}) - \nabla\mathbf{u}(\mathbf{x})\mathbf{x}) \otimes \nabla\phi(\mathbf{x})]\,d\mathbf{x}. \end{aligned} \tag{6.14}$$

for all $\phi \in C_0^1(\Omega;\mathbb{R})$ satisfying (6.7). We claim that (6.14) holds also for all $\phi \in W_0^{1,\infty}(\Omega;\mathbb{R})$ satisfying (6.7). Indeed, for any such ϕ a standard mollification procedure yields a sequence $\{\phi_n\}_{n\geq 1}$ in $C_0^1(\Omega;\mathbb{R}^3)$ satisfying (6.7) and which converges to ϕ weakly* in $W^{1,\infty}(\Omega;\mathbb{R}^3)$.

The main part of the proof consists of taking suitable choices of Lipschitz functions ϕ in (6.14) to show that the following representation of the energy of $\mathbf{u}$ as a boundary integral holds:

$$3E(\mathbf{u}) = \int_{\partial\Omega} W(\nabla\mathbf{u})(\mathbf{x}\cdot N(\mathbf{x})) + \frac{\partial W}{\partial F}(\nabla\mathbf{u}) : [(\mathbf{u}(\mathbf{x}) - \nabla\mathbf{u}(\mathbf{x})\mathbf{x}) \otimes N(\mathbf{x})]\,d\mathcal{H}^2(\mathbf{x}). \tag{6.15}$$

Let $\gamma : [0,1] \to \mathbb{R}$ be a Lipschitz function, and let $\phi : \Omega \to \mathbb{R}$ be the Lipschitz function given by $\phi(\mathbf{x}) := \gamma(|\mathbf{x}|/d(\theta))$, where $\theta = \mathbf{x}/|\mathbf{x}|$. Then, for almost every $\mathbf{x} \in \Omega$,

$$\nabla\phi(\mathbf{x}) = \gamma'\left(\frac{|\mathbf{x}|}{d(\theta)}\right)\left(\frac{1}{d(\theta)}\theta - \frac{|\mathbf{x}|}{d(\theta)^2}A\nabla d(\theta)\right), \tag{6.16}$$

where

$$A := \frac{1}{|\mathbf{x}|}\left(\mathbf{I} - \frac{1}{|\mathbf{x}|^2}\mathbf{x}\otimes\mathbf{x}\right).$$

For $\varepsilon \in (0,1)$, let $\gamma_\varepsilon : [0,1] \to \mathbb{R}$ be given by:

$$\gamma_\varepsilon(\rho) = \begin{cases} 1, & 0 \le \rho \le 1-\varepsilon, \\ (1-\rho)/\varepsilon, & 1-\varepsilon < \rho \le 1. \end{cases} \tag{6.17}$$

Let $\phi_\varepsilon : \Omega \to \mathbb{R}$ be given by $\phi_\varepsilon(\mathbf{x}) = \gamma_\varepsilon(|\mathbf{x}|/d(\theta))$. Then

$$\nabla\phi_\varepsilon(\mathbf{x}) = \begin{cases} 0, & |\mathbf{x}|/d(\theta) < 1-\varepsilon, \\ -\dfrac{1}{\varepsilon}\dfrac{1}{d(\theta)}\left(\theta - (\mathbf{I} - \theta\otimes\theta)\dfrac{\nabla d(\theta)}{d(\theta)}\right), & 1-\varepsilon < |\mathbf{x}|/d(\theta) < 1. \end{cases} \tag{6.18}$$

It follows that

$$\nabla\phi_\varepsilon(\mathbf{x})\cdot\mathbf{x} = \begin{cases} 0, & |\mathbf{x}|/d(\theta) < 1-\varepsilon, \\ -\dfrac{1}{\varepsilon}\dfrac{|\mathbf{x}|}{d(\theta)}, & 1-\varepsilon < |\mathbf{x}|/d(\theta) < 1. \end{cases} \tag{6.19}$$

Taking in (6.14) the particular choice ϕ_ε, for $\varepsilon \in (\alpha, 1)$, and using (6.13), we obtain:

$$\begin{aligned} 3\int_\Omega W(\nabla\mathbf{u}(\mathbf{x}))\phi_\varepsilon(\mathbf{x})\,d\mathbf{x} &= \frac{1}{\varepsilon}\int_{1-\varepsilon}^1 \rho^2\left(\int_{\partial\Omega}\frac{d(\theta)}{\alpha(\theta)}\frac{\rho|\mathbf{x}|}{d(\theta)}W(\nabla\mathbf{u}(\rho\mathbf{x}))\,d\mathcal{H}^2(\mathbf{x})\right)d\rho \\ &+ \frac{1}{\varepsilon}\int_{1-\varepsilon}^1 \rho^2\left(\int_{\partial\Omega}\frac{d(\theta)}{\alpha(\theta)}\left\{\frac{\partial W}{\partial\mathbf{F}}(\nabla\mathbf{u}(\rho\mathbf{x})) : \Big[(\mathbf{u}(\rho\mathbf{x}) - \nabla\mathbf{u}(\rho\mathbf{x})\rho\mathbf{x}) \right.\right. \\ &\left.\left.\otimes\,\frac{1}{d(\theta)}\left(\theta - (\mathbf{I}-\theta\otimes\theta)\frac{\nabla d(\theta)}{d(\theta)}\right)\Big]\right\}d\mathcal{H}^2(\mathbf{x})\right)d\rho \end{aligned} \tag{6.20}$$

Letting $\varepsilon \searrow 0$ in (6.20) and using (6.6) we obtain

$$\begin{aligned} 3E(\mathbf{u}) &= \int_{\partial\Omega}\frac{d(\theta)}{\alpha(\theta)}W(\nabla\mathbf{u}(\mathbf{x}))\,d\mathcal{H}^2(\mathbf{x}) \\ &+ \int_{\partial\Omega}\frac{d(\theta)}{\alpha(\theta)}\left\{\frac{\partial W}{\partial\mathbf{F}}(\nabla\mathbf{u}(\mathbf{x})) : \Big[(\mathbf{u}(\mathbf{x}) - \nabla\mathbf{u}(\mathbf{x})\mathbf{x})\right. \\ &\left.\otimes\,\frac{1}{d(\theta)}\left(\theta - (\mathbf{I}-\theta\otimes\theta)\frac{\nabla d(\theta)}{d(\theta)}\right)\Big]\right\}d\mathcal{H}^2(\mathbf{x}), \end{aligned} \tag{6.21}$$

from where (6.15) now follows upon using (6.11) and (6.12).

Clearly (6.15) also holds if $\mathbf{u}$ is replaced by $\mathbf{u}_\mathbf{A}^{\mathrm{hom}}$. Using this representation of the energy of $\mathbf{u}$ and that of the homogeneous deformation $\mathbf{u}_{\mathbf{A}}^{\mathrm{hom}}$ as boundary integrals, we now show that $E(\mathbf{u}) - E(\mathbf{u}_{\mathbf{A}}^{\mathrm{hom}})$ is negative. Since $\mathbf{u}(\mathbf{x}) = \mathbf{u}_{\mathbf{A}}^{\mathrm{hom}}(\mathbf{x})$ for all $\mathbf{x} \in \partial\Omega$, we obtain

$$
\begin{aligned}
3(E(\mathbf{u})-E(\mathbf{u}_{\mathbf{A}}^{\mathrm{hom}})) &= \int_{\partial\Omega} [W(\nabla\mathbf{u}(\mathbf{x})) - W(\nabla\mathbf{u}_{\mathbf{A}}^{\mathrm{hom}}(\mathbf{x}))](\mathbf{x}\cdot N(\mathbf{x}))\, d\mathcal{H}^2(\mathbf{x}) \\
&\quad + \int_{\partial\Omega} \Big\{ \frac{\partial W}{\partial \mathbf{F}}(\nabla\mathbf{u}(\mathbf{x})) : [(\mathbf{u}_{\mathbf{A}}^{\mathrm{hom}}(\mathbf{x}) - \nabla\mathbf{u}(\mathbf{x})\mathbf{x}) \otimes N(\mathbf{x})] \\
&\qquad - \frac{\partial W}{\partial \mathbf{F}}(\nabla\mathbf{u}_{\mathbf{A}}^{\mathrm{hom}}(\mathbf{x})) : [(\mathbf{u}_{\mathbf{A}}^{\mathrm{hom}}(\mathbf{x}) - \nabla\mathbf{u}_{\mathbf{A}}^{\mathrm{hom}}(\mathbf{x})\mathbf{x}) \otimes N(\mathbf{x})] \Big\}\, d\mathcal{H}^2(\mathbf{x}) \\
&= \int_{\partial\Omega} \Big\{ \big[W(\nabla\mathbf{u}(\mathbf{x})) - W(\nabla\mathbf{u}_{\mathbf{A}}^{\mathrm{hom}}(\mathbf{x}))\big]\, (\mathbf{x}\cdot N(\mathbf{x})) \\
&\qquad - \frac{\partial W}{\partial \mathbf{F}}(\nabla\mathbf{u}(\mathbf{x})) : [(\nabla\mathbf{u}(\mathbf{x})\mathbf{x} - \nabla\mathbf{u}_{\mathbf{A}}^{\mathrm{hom}}(\mathbf{x})\mathbf{x}) \otimes N(\mathbf{x})] \Big\}\, d\mathcal{H}^2(\mathbf{x}) \\
&\quad + \int_{\partial\Omega} \Big\{ \frac{\partial W}{\partial \mathbf{F}}(\nabla\mathbf{u}(\mathbf{x})) : [(\mathbf{u}_{\mathbf{A}}^{\mathrm{hom}}(\mathbf{x}) - \nabla\mathbf{u}_{\mathbf{A}}^{\mathrm{hom}}(\mathbf{x})\mathbf{x}) \otimes N(\mathbf{x})] \\
&\qquad - \frac{\partial W}{\partial \mathbf{F}}(\nabla\mathbf{u}_{\mathbf{A}}^{\mathrm{hom}}(\mathbf{x})) : [(\mathbf{u}_{\mathbf{A}}^{\mathrm{hom}}(\mathbf{x}) - \nabla\mathbf{u}_{\mathbf{A}}^{\mathrm{hom}}(\mathbf{x})\mathbf{x}) \otimes N(\mathbf{x})] \Big\}\, d\mathcal{H}^2(\mathbf{x}).
\end{aligned}
$$

Using Corollary 6.5 it now follows that

$$
\begin{aligned}
&3(E(\mathbf{u}) - E(\mathbf{u}_{\mathbf{A}}^{\mathrm{hom}})) \\
&= \int_{\partial\Omega} \Big\{ W(\nabla\mathbf{u}(\mathbf{x})) - W(\nabla\mathbf{u}_{\mathbf{A}}^{\mathrm{hom}}(\mathbf{x})) - \frac{\partial W}{\partial \mathbf{F}}(\nabla\mathbf{u}(\mathbf{x})) : [\nabla\mathbf{u}(\mathbf{x}) - \nabla\mathbf{u}_{\mathbf{A}}^{\mathrm{hom}}(\mathbf{x})] \Big\} \\
&\qquad\qquad \times (\mathbf{x}\cdot N(\mathbf{x}))\, d\mathcal{H}^2(\mathbf{x}).
\end{aligned}
\tag{6.22}
$$

The required result follows using the rank-one convexity of W (in particular, Lemma 6.4 shows that $\nabla\mathbf{u}$ and $\nabla\mathbf{u}_{\mathbf{A}}^{\mathrm{hom}}$ are rank-one connected matrices) and the fact that $\mathbf{x}\cdot N(\mathbf{x}) > 0$ for all $\mathbf{x} \in \partial\Omega$, which is a consequence of the star-shapedness of Ω.

□

6.2 The weak form of the Green Divergence Identity as a necessary condition for a minimiser

We now turn our attention to obtaining (6.8) as a necessary condition for a local extremum, using ideas from [19]. This paper has as starting point the fact that the weak forms of both the Euler-Lagrange equations (6.4) and energy-momentum equations (6.5) for a (smooth) function $\mathbf{u}$ can be obtained as

$$\frac{d}{dt}E(\mathbf{u}_t)\Big|_{t=0} = 0, \tag{6.23}$$

where $\{\mathbf{u}_t : t \in (-t_0, t_0)\}$ is a family of perturbations of $\mathbf{u}$ with $\mathbf{u}_0 = \mathbf{u}$, namely $\mathbf{u}_t(\mathbf{x}) = \mathbf{u}(\mathbf{x}) + t\varphi(\mathbf{x})$ and respectively $\mathbf{u}_t(\mathbf{x}) = \mathbf{u}(\mathbf{x} + t\varphi(\mathbf{x}))$, for all $\mathbf{x} \in \Omega$, where $\varphi \in C_0^\infty(\Omega; \mathbb{R}^3)$. The paper provides a method for obtaining other necessary conditions for local extrema of variational problems, by choosing more general types of variations $\{\mathbf{u}_t\}$ in (6.23). Here we show that (6.8) is such a necessary condition for local extrema.

Definition 6.6. (i) *A function* $\mathbf{u} \in W^{1,p}(\Omega; \mathbb{R}^3)$ *is a* $W^{1,p}$*-local minimiser for* W *if there exists* $\delta > 0$ *such that* $E(\mathbf{u}) \leq E(\widetilde{\mathbf{u}})$ *for all* $\widetilde{\mathbf{u}} \in W^{1,p}(\Omega; \mathbb{R}^3)$ *such that* $\widetilde{\mathbf{u}} = \mathbf{u}$ *on* $\partial\Omega$ *and* $||\widetilde{\mathbf{u}} - \mathbf{u}||_{W^{1,p}(\Omega;\mathbb{R}^3)} < \delta$.

(ii) *A function* $\mathbf{u} \in W^{1,p}(\Omega; \mathbb{R}^3)$ *is a* $W^{1,p}$*-local maximiser for* W *if there exists* $\delta > 0$ *such that* $E(\mathbf{u}) \geq E(\widetilde{\mathbf{u}})$ *for all* $\widetilde{\mathbf{u}} \in W^{1,p}(\Omega; \mathbb{R}^3)$ *such that* $\widetilde{\mathbf{u}} = \mathbf{u}$ *on* $\partial\Omega$ *and* $||\widetilde{\mathbf{u}} - \mathbf{u}||_{W^{1,p}(\Omega;\mathbb{R}^3)} < \delta$.

Given $\phi \in C_0^1(\Omega; \mathbb{R})$, consider the family of mappings $\{\psi_t\}_{t\in(-t_0,t_0)}$ given by

$$\psi_t(\mathbf{x}) = (1 + t\phi(\mathbf{x}))\mathbf{x}, \tag{6.24}$$

where $t_0 > 0$ is sufficiently small so that, for all $t \in (-t_0, t_0)$,

$$\det \nabla\psi_t > 0 \quad \text{for all } \mathbf{x} \in \Omega. \tag{6.25}$$

Since, for any $t \in (-t_0, t_0)$, ψ_t belongs to $C^1(\Omega; \mathbb{R}^3) \cap C(\overline{\Omega}; \mathbb{R}^3)$, coincides with the identity on $\partial\Omega$ and satisfies (6.25), [14, Theorem 5.5-2, p.225] shows that ψ_t is a diffeomorphism of Ω onto itself and a homeomorphism from $\overline{\Omega}$ onto itself.

Thus, a family of variations of $\mathbf{u}$ can be defined by setting

$$\begin{cases} \mathbf{u}_t(\mathbf{z}) = (1 + t\phi(\mathbf{x}))\mathbf{u}(\mathbf{x}), \\ \mathbf{z} = \psi_t(\mathbf{x}). \end{cases} \tag{6.26}$$

Theorem 6.7. *Let* $\mathbf{A} \in M_+^{3\times 3}$ *and let* $\mathbf{u} \in W^{1,1}(\Omega;\mathbb{R}^3)$ *satisfy* (6.6) *and* $E(\mathbf{u}) < \infty$, $\mathbf{u}(\mathbf{x}) = \mathbf{A}\mathbf{x}$ *on* $\partial\Omega$, *be such that, for every* $\phi \in C_0^1(\Omega;\mathbb{R})$ *with the property* (6.7), *the family of variations* $\{\mathbf{u}_t\}_{t\in(-t_0,t_0)}$ *given by* (6.26) *satisfies* (6.23). *Then* (6.8) *holds. In particular, if* $\mathbf{u} \in W^{1,p}(\Omega;\mathbb{R}^3)$, *for some* $p \in [1,\infty)$, *satisfies* (6.6), $\mathbf{u}(\mathbf{x}) = \mathbf{A}\mathbf{x}$ *on* $\partial\Omega$ *and is a* $W^{1,p}$*-local extremum of* W, *then* (6.8) *holds.*

Proof of Theorem 6.7. We start by explaining how the second part of the theorem follows from the first. Let $\mathbf{u} \in W^{1,p}(\Omega;\mathbb{R}^3)$, for some $p \in [1,\infty)$, such that (6.6) holds and $\mathbf{u}(\mathbf{x}) = \mathbf{A}\mathbf{x}$ on $\partial\Omega$ be a $W^{1,p}$-local extremum of W. Let the family of variations $\{\mathbf{u}_t\}_{t\in(-t_0,t_0)}$ be given by (6.26). Then

$$\nabla\mathbf{u}_t(\mathbf{z}) = [(1 + t\phi(\mathbf{x}))\nabla\mathbf{u}(\mathbf{x}) + t\mathbf{u}(\mathbf{x}) \otimes \nabla\phi(\mathbf{x})][(1 + t\phi(\mathbf{x}))\mathbf{I} + t\mathbf{x} \otimes \nabla\phi(\mathbf{x})]^{-1},$$
$$\text{where } \mathbf{x} = \psi_t^{-1}(\mathbf{z}). \tag{6.27}$$

It follows from (6.27) that $\mathbf{u}_t$ belongs to $W^{1,p}(\Omega;\mathbb{R}^3)$ and, since $\mathbf{u}$ satisfies (6.6) and ϕ satisfies (6.7), it follows that $\det\nabla\mathbf{u}_t > 0$ almost everywhere, for all t sufficiently small. (Note that, if ϕ were not assumed to satisfy (6.7), then it would not be possible to deduce, irrespective of how small t is, that $\det\nabla\mathbf{u}_t > 0$ almost everywhere, a condition which is necessary for $\mathbf{u}_t$ to have finite energy.) It is easy to check that

$$\mathbf{u}_t \to \mathbf{u} \text{ in } W^{1,p}(\Omega;\mathbb{R}^3) \text{ as } t \to 0.$$

Thus, if $\mathbf{u}$ is a $W^{1,p}$-local extremum of W, then (6.23) necessarily holds, provided that the mapping $t \mapsto E(\mathbf{u}_t)$ is differentiable at $t = 0$. We shall see later that this differentiability requirement is not an issue.

We now prove the first part of the theorem. It follows from (6.27) and the change of variables formula that, for all t sufficiently small,

$$
\begin{aligned}
&E(\mathbf{u}_t) = \\
&= \int_\Omega W([(1+t\phi(\mathbf{x}))\nabla\mathbf{u}(\mathbf{x}) + t\mathbf{u}(\mathbf{x}) \otimes \nabla\phi(\mathbf{x})][(1+t\phi(\mathbf{x}))\mathbf{I} + t\mathbf{x} \otimes \nabla\phi(\mathbf{x})]^{-1}) \\
&\qquad\qquad \times \det[(1+t\phi(\mathbf{x}))\mathbf{I} + t\mathbf{x} \otimes \nabla\phi(\mathbf{x})]\, d\mathbf{x} \\
&=: \int_\Omega f_t(\mathbf{x})\, d\mathbf{x} \\
&= \int_{\tau\Omega} f_t(\mathbf{x})\, d\mathbf{x} + \int_{\Omega\setminus\tau\Omega} f_t(\mathbf{x})\, d\mathbf{x}.
\end{aligned} \tag{6.28}
$$

Let $c \in \mathbb{R}$ be such that $\phi(\mathbf{x}) = c$ for all $\mathbf{x} \in \tau\Omega$. Then

$$\int_{\tau\Omega} f_t(\mathbf{x})\, d\mathbf{x} = (1+ct)^3 \int_{\tau\Omega} W(\nabla\mathbf{u}(\mathbf{x}))\, d\mathbf{x}, \tag{6.29}$$

so that

$$\frac{d}{dt}\left(\int_{\tau\Omega} f_t(\mathbf{x})\, d\mathbf{x}\right)\Big|_{t=0} = 3c \int_{\tau\Omega} W(\nabla\mathbf{u}(\mathbf{x}))\, d\mathbf{x}. \tag{6.30}$$

On the other hand, since $\mathbf{u}$ satisfies (6.6), there is no difficulty in justifying the differentiation under the integral sign, which yields

$$
\begin{aligned}
\frac{d}{dt}\left(\int_{\Omega\setminus\tau\Omega} f_t(\mathbf{x})\, d\mathbf{x}\right)\Big|_{t=0} &= \int_{\Omega\setminus\tau\Omega} W(\nabla\mathbf{u}(\mathbf{x}))[3\phi(\mathbf{x}) + \nabla\phi(\mathbf{x}) \cdot \mathbf{x}]\, d\mathbf{x} \\
&\quad + \int_{\Omega\setminus\tau\Omega} \frac{\partial W}{\partial \mathbf{F}}(\nabla\mathbf{u}(\mathbf{x})) : [(\mathbf{u}(\mathbf{x}) - \nabla\mathbf{u}(\mathbf{x})\mathbf{x}) \otimes \nabla\phi(\mathbf{x})]\, d\mathbf{x}.
\end{aligned} \tag{6.31}
$$

Observe now that, since $\phi(\mathbf{x}) = c$ for all $\mathbf{x} \in \tau\Omega$, one can write (6.30) as

$$
\begin{aligned}
\frac{d}{dt}\left(\int_{\tau\Omega} f_t(\mathbf{x})\, d\mathbf{x}\right)\Big|_{t=0} &= \int_{\tau\Omega} W(\nabla\mathbf{u}(\mathbf{x}))[3\phi(\mathbf{x}) + \nabla\phi(\mathbf{x}) \cdot \mathbf{x}]\, d\mathbf{x} \\
&\quad + \int_{\tau\Omega} \frac{\partial W}{\partial \mathbf{F}}(\nabla\mathbf{u}(\mathbf{x})) : [(\mathbf{u}(\mathbf{x}) - \nabla\mathbf{u}(\mathbf{x})\mathbf{x}) \otimes \nabla\phi(\mathbf{x})]\, d\mathbf{x}.
\end{aligned} \tag{6.32}
$$

The preceding considerations show that the mapping $t \mapsto E(\mathbf{u}_t)$ is differentiable at $t = 0$, and

$$\frac{d}{dt}E(\mathbf{u}_t)\Big|_{t=0} = \int_\Omega W(\nabla\mathbf{u}(\mathbf{x}))[3\phi(\mathbf{x}) + \nabla\phi(\mathbf{x})\cdot\mathbf{x}]\,d\mathbf{x}$$
$$+\int_\Omega \frac{\partial W}{\partial \mathbf{F}}(\nabla\mathbf{u}(\mathbf{x})) : [(\mathbf{u}(\mathbf{x}) - \nabla\mathbf{u}(\mathbf{x})\mathbf{x}) \otimes \nabla\phi(\mathbf{x})]\,d\mathbf{x}. \tag{6.33}$$

Since (6.23) holds, we conclude upon re-arranging the terms in (6.33) that (6.8) is satisfied. This completes the proof of Theorem 6.7.

□

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